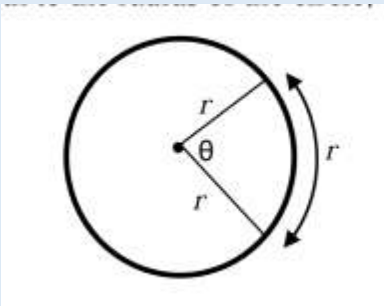
MASTERMIND SCHOLARS EDUCATION ALLIANCE

ADVANCED MATHEMATICS
TRIGONOMETRY

Radian Measure in calculus and advanced trigonometry, angles are usually measured in radians. One radian is the measure of the angle at the center of the circle subtended by an arc equal in length to the radius of the circle.



Relationship Between Radians and Degrees

Radians are often called "circular measure" and are denoted by rads. The number of radians in one complete revolution is given by the ratio:

$$\frac{Circumference\ of\ the\ circle}{radius}$$

$$\frac{2\pi r}{r}$$

2π radian

One complete revolution = 2π radian

= 360^o

$$\pi\ radian = 180^0$$

Table memorization on the relationship between radian and degree measure

Angle in degree	Radian equivalent
0	0
30	$\frac{\pi}{6}$
45	$\frac{\pi}{4}$
60	$\frac{\pi}{3}$
90	$\frac{\pi}{2}$
120	$\frac{2\pi}{3}$
135	$\frac{3\pi}{4}$

Example 1

Convert 330^0 into radian

Solution

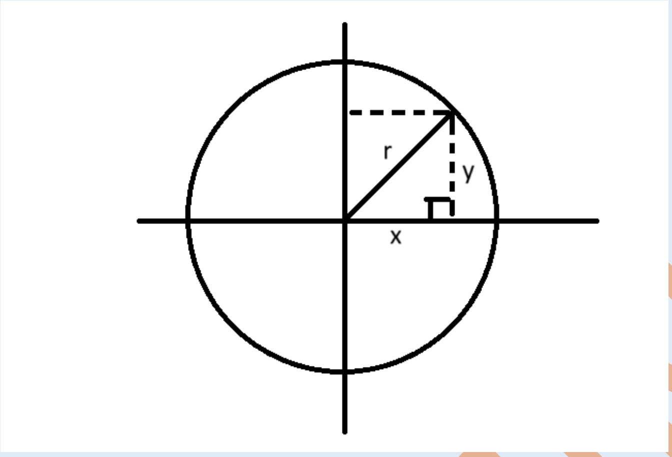
$$\pi\ radian = 180^0$$

$$x = 330o$$

$$330\,\pi = 180\,x$$

$$\frac{330\pi}{180} = x$$

$$x = \frac{11\pi}{6}$$



SOHCAHTOA

Writing out the three trigonometry basic ratios in terms of the diagram above

$\text{Sin}\theta = \frac{opposite}{hypothenus}$ $\text{Sin }\theta = \frac{y}{r}$

$\text{Cos}\theta = \frac{adjacent}{hypothenus}$ $\text{Cos}\theta = \frac{x}{r}$

$\tan\theta = \frac{opposite}{adjacent}$ $\tan\theta = \frac{y}{x}$

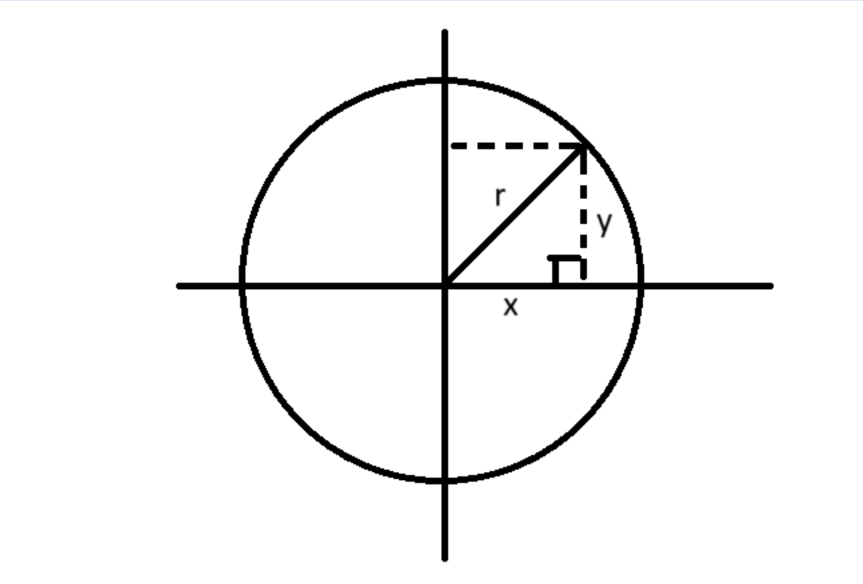
The three other trigonometry ratios which are the reciprocal of the three main once

$\frac{1}{sin\theta}$ = Cosecant θ (cosec θ) $= \frac{1}{\frac{y}{r}} = \frac{r}{y}$

$\frac{1}{Cos\theta} = secant\ \theta = \frac{1}{\frac{x}{r}} = \frac{r}{x}$

$\frac{1}{tan\theta} = cotangent\ \theta\ (\cot\theta) = \frac{1}{\frac{y}{x}} = \frac{x}{y}$

Deriving the three trigonometric equations or identity



from Pythagoras theorem $x^2 + y^2 = r^2$

To obtain the first trigonometric equation we divide the above equation by r^2

$$x^2 + y^2 = r^2$$

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{r^2}{r^2}$$

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

$cos^2\theta + sin^2\theta = 1$ ---------------- 1st trig identity

Derive the second trigonometric identity

To obtain the second trigonometric equation we divide the above equation by y^2

$$x^2 + y^2 = r^2$$

$$\frac{x^2}{y^2} + \frac{y^2}{y^2} = \frac{r^2}{y^2}$$

$$\left(\frac{x}{y}\right)^2 + \left(\frac{y}{y}\right)^2 = \left(\frac{r}{y}\right)^2$$

$cot^2\theta + 1 = cosec^2\theta$ ----------- second trigonometric identity

Deriving the third Trigonometric identity

To obtain the third trigonometric equation we divide the above equation by x^2

$$x^2 + y^2 = r^2$$

$$\frac{x^2}{x^2} + \frac{y^2}{x^2} = \frac{r^2}{x^2}$$

$$\left(\frac{x}{x}\right)^2 + \left(\frac{y}{x}\right)^2 = \left(\frac{r}{x}\right)^2$$

$1 + tan^2\theta = sec^2\theta$ ------------ third trig identity

We obtain the three trigonometric identities as

1. $cos^2\theta + sin^2\theta$ = 1 ------------ first trigonometric identity

a. $cos^2\theta$ = 1 $- sin^2\theta$

b. $sin^2\theta$ = 1 $- cos^2\theta$

2. $cot^2\theta$ + 1 = $cosec^2\theta$ -------------- second trigonometric identities

a. $cot^2\theta$ = $-1 + cosec^2\theta$

b. $cosec^2\theta - cot^2\theta$ = 1

3. 1 + $tan^2\theta$ = $sec^2\theta$ ---------------- third trigonometric identity

a. $tan^2\theta$ = $-1 + sec^2\theta$

b. $tan^2\theta - sec^2\theta = -1$

4. tan θ = $\frac{\sin\theta}{\cos\theta}$

In trigonometry, positive angles are measured in anticlockwise direction from the positive x – direction , starting from 0 to 360

Sin $\theta = \frac{y}{r}$ cos $\theta = -\frac{x}{r}$ tan $\theta = -\frac{y}{x}$	Sin +	All positive	Sin $\theta = \frac{y}{r}$ cos $\theta = \frac{x}{r}$ tan $\theta = \frac{y}{x}$
Sin $\theta = \frac{-y}{r}$ cos $\theta = \frac{-x}{r}$ tan $\theta = \frac{-y}{-x} = \frac{y}{x}$	Tan +	Cos +	Sin $\theta = \frac{-y}{r}$ cos $\theta = \frac{x}{r}$ tan $\theta = \frac{-y}{x}$

Identification of the various quadrants in trigonometry

II Second Quadrant	I First Quadrant
III	IV

Third Quadrant Fourth Quadrant

Memorize the summary of the ratios of the special angles.

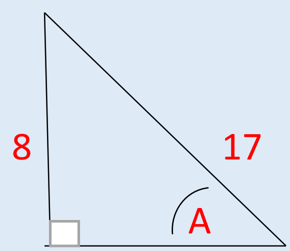
	0	30°	45°	60°	90°
Sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tangent	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Undefined

Example 1

1. Given that Sin A = $\frac{8}{17}$ and that A is an obtuse angle, find cos A and tan A without using tables.

Solution

Using pythagoras theorem



x

$$17^2 = 8^2 + x^2$$
$$17^2 - 8^2 = x^2$$
$$x = \sqrt{17^2 - 8^2}$$
$$x = \sqrt{289 - 64}$$
$$x = \sqrt{225}$$
$$x = 15$$

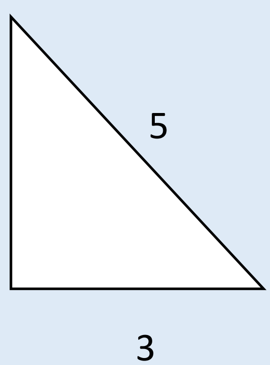
Sin A= $\frac{8}{17}$ Cos$A$ = $-\frac{15}{17}$ tan$A$ = $-\frac{8}{15}$

Alternative Method

Example 2

If 270 < A < 360 and cos A = 0.6 find sin A and tan A without using tables

Solution



Cos A = 0.6 = $\frac{6}{10}$ = $\frac{3}{5}$

Pythagoras theorem

$$5^2 = 3^2 + x^2$$

$$5^2 - 3^2 = x^2$$

$$x = \sqrt{5^2 - 3^2}$$

$$x = \sqrt{25 - 9}$$

$$x = \sqrt{16}$$

$$x = 4$$

Sin A= $-\frac{4}{5}$ (-0.8) Cos$A$ = $\frac{3}{5}$ tan$A$ = $-\frac{4}{3}$ (-1.33)

Alternative Method

Simplifying or establishing trigonometric identities

Example 1

Simplify

$$\frac{1}{1-\cos\theta}+\frac{1}{1+\cos\theta}$$

$$\frac{(1+\cos\theta)+(1-\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}$$

$$\frac{1+\cos\theta+1-\cos\theta}{(1-\cos\theta)(1+\cos\theta)}$$

$$\frac{2}{(1-\cos\theta)(1+\cos\theta)}$$

$$\frac{2}{1+\cos\theta-\cos\theta\ -\ cos^2\theta}$$

$$\frac{2}{1-\ cos^2\theta}$$

but form the first trigonometric identity $sin^2\theta$ = 1 - $cos^2\theta$

$$\frac{2}{sin^2\theta}$$

Simplify

$$\frac{1}{1-\sin\theta}+\frac{1}{1+\sin\theta}$$

$$\frac{(1+\sin\theta)+(1-\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}$$

$$\frac{1+\sin\theta+1-\sin\theta}{(1-\sin\theta)(1+\sin\theta)}$$

$$\frac{2}{(1-\sin\theta)(1+\sin\theta)}$$

$$\frac{2}{1+\sin\theta-\sin\theta\ -\ sin^2\theta}$$

$$\frac{2}{1-\ sin^2\theta}$$

but form the first trigonometric identity $cos^2\theta$ = 1 - $sin^2\theta$

$$\frac{2}{cos^2\theta}$$

Example 3

Simplify $\frac{\text{Sec}\,\theta}{\cos\theta} = \frac{\tan\theta}{\cot\theta}$

Solution

$$\frac{\text{Sec}\,\theta}{\cos\theta} - \frac{\tan\theta}{\cot\theta}$$

But $\text{Sec}\,\theta = \frac{1}{\cos\theta}$ $\quad \tan\theta = \frac{\sin\theta}{\cos\theta}$ $\quad \cot\theta = \frac{1}{\tan\theta}$

$$\frac{\frac{1}{\cos\theta}}{\cos\theta} - \frac{\tan\theta}{\frac{1}{\tan\theta}}$$

$$\left(\frac{1}{\cos\theta} \div \cos\theta\right) - \left(\tan\theta \div \frac{1}{\tan\theta}\right)$$

$$\left(\frac{1}{\cos\theta} \times \frac{1}{\cos\theta}\right) - (\tan\theta \times \tan\theta)$$

$$\frac{1}{cos^2\theta} - tan^2\theta$$

$$sec^2\theta - tan^2\theta$$

$$1$$

From the second trigonometric identity 1 $= sec^2\theta - tan^2\theta$

Example 4

Show that

$$\cot\theta + \frac{\sin\theta}{1+\cos\theta} = \text{cosec}\,\theta$$

$$\frac{1}{\tan\theta} + \frac{\sin\theta}{1+\cos\theta}$$

$$\frac{1}{\tan\theta} + \frac{\sin\theta}{1+\cos\theta}$$

But $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$$\frac{1}{\frac{\sin\theta}{\cos\theta}} + \frac{\sin\theta}{1+\cos\theta}$$

$$\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{1+\cos\theta}$$

$$\frac{(1+\cos\theta)(\cos\theta) + (\sin\theta)(\sin\theta)}{(\sin\theta)(1+\cos\theta)}$$

$$\frac{\cos\theta + cos^2\theta + sin^2\theta}{(\sin\theta)(1+\cos\theta)}$$

From the first trig identity $cos^2\theta + sin^2\theta = 1$

$$\frac{\cos\theta + 1}{(\sin\theta)(1 + \cos\theta)}$$

$$\frac{1 + \cos\theta}{(\sin\theta)(1 + \cos\theta)}$$

$$\frac{1}{\sin\theta}$$

$$\operatorname{cosec}\theta$$

Example

Given that $\tan\theta = \frac{3}{4}$ find the value of $\frac{sin\theta}{cos\theta + sin\theta}$

Example

θ is an obtuse angle and $\sin\theta = 0.6$. Without using tables, find the value of

i. $1 - \cos\theta$

ii. $\frac{2\tan\theta}{1 - tan^2\theta}$

EXAMPLE 5

Tanθ + cotθ = $\frac{1}{sin\theta cos\theta}$

Example 6

Prove that $\frac{\tan\theta}{1+ tan^2\theta} = sin\theta cos\theta$

Example 7

1. Prove that $\frac{\cot A-1}{\cot A+1}$ = $\frac{1-\tan A}{1+\tan A}$

Example 8

2. Prove that $\sqrt{\frac{1-\sin A}{1+\tan A}}$ = secA – tan A

Example 9

3. Given that tan θ = $\frac{3}{4}$ find the values of $\frac{\sin\theta}{\cos\theta+\sin\theta}$

Example 10

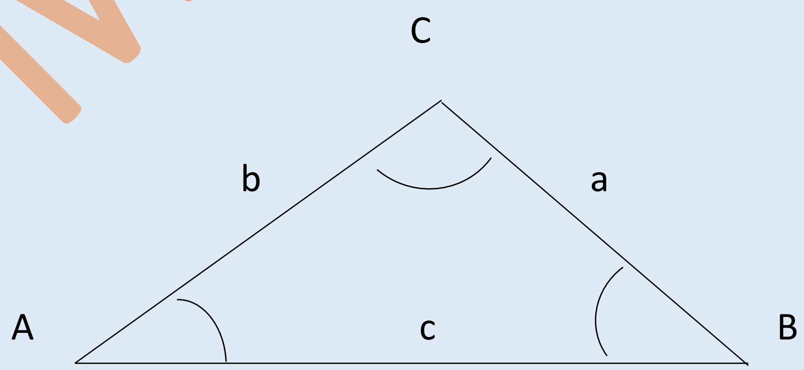
4. Show that $\frac{1}{\sec A-\tan A}$ = sec A + tan A

Example 11

5. Show that $\frac{Cos\ A}{1-\sin A}$ = $\frac{1+\sin A}{\cos A}$

Using Sine and Cosine Rules

Sine rules



$$\frac{a}{sina} = \frac{b}{sinb} = \frac{c}{sinc}$$

When to Use the Sine Rule

Use it when you know **a side and its opposite angle**.

It works in two main cases:

1□. **ASA / AAS**
You know **2 angles + 1 side** → find the missing side(s).

2□. **SSA (Ambiguous case)**
You know **2 sides + a non-included angle** → find another side or angle. (A **non-included angle** is an angle that is **NOT between the two known sides** of a triangle.)

The cosine rule

$$a^2 = b^2 + c^2 - 2abcosA$$