MASTERMIND SCHOLARS EDUCATIONAL CENTRE

ADVANCED MATHEMATICS

BINOMIAL PROBABILITY

In our previous Mastermind classes, we learned that **a combination** is the selection of a number of objects **when the order of arrangement is not important**. In other words, we only care about *which items are chosen*, not the order in which they appear.

When we apply combinations to **probability**, we are usually dealing with situations where a certain number of items are drawn **at random** from a bag, box, or group. These selections may be done **with replacement** or **without replacement**, depending on the question.

Understanding combinations is very important in probability because many real-life probability problems involve **counting the number of possible selections** and comparing them with the number of **favourable selections**.

We will now explore how combinations are used to solve **simple probability problems**, especially in situations involving random selection.

Example 1

A bag contain 30 balls which are identical except for colour; 12 of the balls are red and 18 are blue. 4 balls are picked at random out of the bag. What is the probability that

i. they are all red ?

ii. they are of the same colour ?

iii. two are red and two are blue ?

solution

n(red balls) = 12

n(blue balls) = 18

n(Total balls) = 30

4 balls are picked at random

i. P(all 4 balls are red) = $\frac{number\ of\ ways\ of\ picking\ 4\ red\ out\ of\ 12}{number\ of\ ways\ of\ picking\ 4\ balls\ out\ of\ 30}$

$$\frac{12C_4}{30C_4}$$

$$\frac{11}{609}$$

ii. P(all 4 balls are of the same colour) = P(all 4 are red) or P(all 4 are blue)

$$\frac{number\ of\ ways\ of\ picking\ 4\ red\ balls\ out\ of\ 12\ or\ 4\ blue\ balls\ out\ of\ 18}{Number\ of\ ways\ of\ picking\ 4\ balls\ out\ of\ 30}$$

$$\frac{12C_4 + 18C_4}{30C_4}$$

$$\frac{79}{609}$$

iii. P(two red and two blue balls) = P(all 2 are red) and P(all 2 are blue)

$$\frac{number\ of\ ways\ of\ picking\ 2\ red\ balls\ out\ of\ 12\ and\ 2\ blue\ balls\ out\ of\ 18}{Number\ of\ ways\ of\ picking\ 4\ balls\ out\ of\ 30}$$

$$\frac{12C_2 \times 18C_2}{30C_4}$$

$$\frac{374}{1015}$$

Example 2

A box contain 7 red, 8 white and 3 blue balls. If three balls are drawn from the box, calculate the probability that;

i. two are red and one is blue;

ii. at least two are white;

iii. one of each colour.

Answer

n(red balls) = 7

n(white balls) = 8

n(blue balls) = 3

n(Total) = 18

3 balls are picked at random

i. P(two are red and one is blue) $= \frac{number\ of\ ways\ of\ picking\ 2\ red\ out\ of\ 7\ and\ 1\ blue\ out\ of\ 3}{number\ of\ ways\ of\ picking\ 3\ balls\ out\ of\ 18}$

$$\frac{7C_2 \times 3C_1}{18C_3}$$

$$\frac{21 \times 3}{816}$$

$$\frac{21}{272}$$

ii. P(at least 2 white) =

conditions

1. two white balls and 1 blue ball

2. two white ball and 1 red ball

3. 3 white ball

ii. P(at least two white balls) = P(two white and 1 blue) or P(two white and 1 red) or P(3 white)

$$\frac{P(two\ white\ and\ 1\ blue)\ or\ P(two\ white\ and\ 1\ red)\ or\ P(3\ white)}{Number\ of\ ways\ of\ picking\ 3\ balls\ out\ of\ 18}$$

$$\frac{[8C_2 \times 3C_1] + [8C_2 \times 7C_1] + [8C_3]}{18C_3}$$

$$\frac{84 + 196 + 56}{816}$$

$$\frac{336}{816}$$

$$\frac{7}{17}$$

iii. P(one of each colour) = P(one white) and P (one blue) and P(one red)

$$\frac{P(one\ white)\ and\ P(one\ blue)\ and\ P(one\ red)}{Number\ of\ ways\ of\ picking\ 3\ balls\ out\ of\ 18}$$

$$\frac{[8C_1] \times [3C_1] \times [7C_1]}{18C_3}$$

$$\frac{8 \times 3 \times 7}{18C_3}$$

$$\frac{168}{816}$$

$$\frac{7}{34}$$

Example 3

A bag contains 20 small identical objects, 8 of them are black, 7 are red and the rest are white. If three of the objects are selected at random from the bag at once, find the probability that;

i. one is black , one is red and the remaining is white

ii. Exactly two are red

iii. None of them is white

Solution

n(black balls)= 8

n(red balls) = 7

n(white balls)= 5

n(total balls) = 20

3 balls are chosen

i. one is black , one is red and the remaining is white

P(one of each colour) = P(one black) and P (one red) and P(one white)

$$\frac{P(one\ black)\ and\ P(one\ red)\ and\ P(one\ white)}{Number\ of\ ways\ of\ picking\ 3\ balls\ out\ of\ 20}$$

$$\frac{[8C_1] \times [7C_1] \times [5C_1]}{20C_3}$$

$$\frac{8 \times 7 \times 5}{20C_3}$$

$$\frac{280}{1140}$$

$$\frac{14}{57}$$

ii. P(exactly 2 are red) =

1. 2 red balls and 1 black ball

2. 2 red balls and 1 white ball

P(exactly two red balls) = P(two red and 1 black) or P(two red and 1 white)

$$\frac{p(two\ red\ and\ 1\ black) + P(two\ red\ and\ 1\ white)}{Number\ of\ ways\ of\ picking\ 3\ balls\ out\ of\ 20}$$

$$\frac{[7C_2 \times\ 8C_1\ ] +\ [7C_2 \times\ 5C_1\ ]}{20C_3}$$

$$\frac{168\ + 105}{1140}$$

$$\frac{273}{1140}$$

$$\frac{91}{380}$$

Alternative method for P(exactly 2 red)

P(exactly 2 red balls) = P (2 red balls and one of other colour)

$$\frac{P(two\ red\ and\ 1\ of\ other\ colour)}{Number\ of\ ways\ of\ picking\ 3\ balls\ out\ of\ 20}$$

$$\frac{[7C_2 \times\ 13C_1\ ]}{20C_3}$$

$$\frac{21 \times\ 13}{1140}$$

$$\frac{273}{1140}$$

$$\frac{91}{380}$$

iii. P(None is white) = P(3 non white balls)

$$\frac{P(3\ non\ white\ balls)}{Number\ of\ ways\ of\ picking\ 3\ balls\ out\ of\ 20}$$

$$\frac{[\ 15C_3\ ]}{20C_3}$$

$$\frac{455}{1140}$$

$$\frac{91}{228}$$

Example 4

A bag contains 8 red , 3 white and 9 black balls. If three are drawn at random, determine the probability that

i. all 3 are red

ii. all 3 are white

iii. 2 are red and 1 white

iv. at least 1 is white

Solution

n(black balls)= 9

n(red balls) = 8

n(white balls)= 3

n(total balls) = 20

3 balls are chosen

i. all three are red

P(all three are red) = $\frac{P(picking\ three\ red\ balls\ out\ of\ 8)}{Number\ of\ ways\ of\ picking\ 3\ balls\ out\ of\ 20}$

$$\frac{[8C_3\ ]}{20C_3}$$

$$\frac{56}{1140}$$

$$\frac{14}{285}$$

ii. all three are white

P(all three are white) = $\frac{P(picking\ three\ white\ balls\ out\ of\ 3)}{Number\ of\ ways\ of\ picking\ 3\ balls\ out\ of\ 20}$

$$\frac{[3C_3\ ]}{20C_3}$$

$$\frac{1}{1140}$$

iii. 2 are red and 1 is white

P(2 red and 1 white) = $\frac{P(picking\ 2\ red\ balls\ out\ of\ 8\ and\ 1\ white\ out\ of\ 3)}{Number\ of\ ways\ of\ picking\ 3\ balls\ out\ of\ 20}$

$$\frac{[8C_2\ \times\ 3C_1]}{20C_3}$$

$$\frac{28\ \times 3}{1140}$$

$$\frac{84}{285}$$

$$\frac{28}{95}$$

iv. at least 1 is white

P(at least 1 is white)

Conditions

1 white 2 of other colour

2 white one of other colour

3 white

$$\frac{P(1\ white\ and\ 2\ of\ other\ colour)\ or\ P(two\ white\ and\ 1\ of\ other\ colour)\ or\ P(3\ white)}{Number\ of\ ways\ of\ picking\ 3\ balls\ out\ of\ 20}$$

$$\frac{[3C_1 \times\ 17C_2\ ] +\ [3C_2 \times\ 17C_1\ ] + [3C_3\ ]}{20C_3}$$

$$\frac{[3 \times\ 136\ ] +\ [3 \times\ 17\ ] + 1}{1140}$$

$$\frac{408 + 51 + 1}{1140}$$

$$\frac{23}{57}$$

EXAMPLE 5

In a class of 12 students 9 are boys. If two students are selected at random, what is the probability that they are

i. both girls

ii. a boy and a girl

Solution

Try this out

BINOMIAL PROBABILITY CONCEPT

A **binomial situation** occurs when an experiment has **only two mutually exclusive outcomes**. This simply means each trial results in either:

- **Success** or
- **Failure**

Examples: Yes/No, Pass/Fail, Head/Tail.

Notation Used

In binomial probability:

- The probability of **success** is denoted by **p**
- The probability of **failure** is denoted by **q**

Since success and failure are the only possible outcomes, their probabilities must add up to 1:

$$p + q = 1$$

So we can write:

$$p = 1 - q \quad \text{OR} \quad q = 1 - p$$

Binomial Probability Formula

The probability of getting exactly **x successes** in **n independent trials** is given by:

$$P(X = x) = nC_x \, P^x q^{n-x}$$

Where:

- **n** = total number of independent trials
- **x** = number of successes required
- **p** = probability of success in one trial
- **q** = probability of failure in one trial

EXAMPLE 1

A fair coin is tossed six times. Find the probability of getting

i. exactly two heads

ii. at least to heads

iii. two tails

iv. at most two heads

Solution

n = number of independent trials = 6

$P = \frac{1}{2}$

$q = 1 - \frac{1}{2} = \frac{1}{2}$

$$P(X = x) = nC_x P^x q^{n-x}$$

$$x = 2$$

$$P(X = 2) = 6C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{6-2}$$

$$P(X = 2) = 15\left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^4$$

$$P(X = 2) = 15 \times \frac{1}{4} \times \frac{1}{16}$$

$$P(X = 2) = 15 \times \frac{1}{64}$$

$$P(X = 2) = \frac{15}{64}$$

ii. P (at least 2 heads) = P(two or more heads)

= P(X ≥ 2)

There are two ways of resolving this

Method 1: Using the normal method. (Tedious)

$$P(X \geq 2) = P(X = 2) + P(X = 3) + P(X = 4) + P (X = 5) + P (X = 6)$$

Let humbly go through the pain and find out values of each outcome…

$x = 2$

$$P(X = 2) = 6C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{6-2}$$

$$P(X = 2) = 15\left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^4$$

$$P(X = 2) = 15 \times \frac{1}{4} \times \frac{1}{16}$$

$$P(X = 2) = 15 \times \frac{1}{64}$$

$$P(X = 2) = \frac{15}{64}$$

$x = 3$

$$P(X = 3) = 6C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{6-3}$$

$$P(X = 3) = 20\left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^3$$

$$P(X = 3) = 20 \times \frac{1}{8} \times \frac{1}{8}$$

$$P(X = 3) = 20 \times \frac{1}{64}$$

$$P(X = 3) = \frac{5}{16}$$

$x = 4$

$$P(X = 4) = 6C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^{6-4}$$

$$P(X = 4) = 15\left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^2$$

$$P(X = 4) = 15 \times \frac{1}{16} \times \frac{1}{4}$$

$$P(X = 4) = 15 \times \frac{1}{64}$$

$$P(X = 4) = \frac{15}{64}$$

$x = 5$

$$P(X = 5) = 6C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^{6-5}$$

$$P(X = 5) = 6\left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^1$$

$$P(X = 5) = 6 \times \frac{1}{32} \times \frac{1}{2}$$

$$P(X = 5) = 6 \times \frac{1}{64}$$

$$P(X = 5) = \frac{3}{32}$$

$x = 6$

$$P(X = 6) = 6C_6 \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^{6-6}$$

$$P(X = 6) = 1\left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^0$$

$$P(X = 6) = 1 \times \frac{1}{64}$$

$$P(X = 6) = 1 \times \frac{1}{64}$$

$$P(X = 6) = \frac{1}{64}$$

$$P(X \geq 2) = P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6)$$

$$P(X \geq 2) = \frac{15}{64} + \frac{5}{16} + \frac{15}{64} + \frac{3}{32} + \frac{1}{64}$$

$$P(X \geq 2) = \frac{57}{64}$$

In general, this approach is tedious and not advisable to use

Note: For 'at least' and 'at most' with large ranges, always prefer the complement method to save time.

Method 2 : Solve the using the complement method

$$P(X \geq 2)(not\ more\ than\ 2) = 1 - [P(X = 0) + P(X = 1)]$$

$x = 0$

$$P(X = 0) = 6C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{6-0}$$

$$P(X = 0) = 1\left(\frac{1}{2}\right)^6$$

$$P(X = 0) = 1 \times \frac{1}{64}$$

$$P(X = 0) = 1 \times \frac{1}{64}$$

$$P(X = 0) = \frac{1}{64}$$

$x = 1$

$$P(X = 1) = 6C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{6-1}$$

$$P(X = 1) = 6\left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^5$$

$$P(X = 1) = 6 \times \frac{1}{2} \times \frac{1}{32}$$

$$P(X = 1) = 6 \times \frac{1}{64}$$

$$P(X = 1) = \frac{3}{32}$$

$$P(X \geq 2)(not\ more\ than\ 2) = 1 - [P(X = 0) + P(X = 1)]$$

$$P(X \geq 2)(not\ more\ than\ 2) = 1 - (\frac{1}{64} + \frac{3}{32})$$

$$1 - \frac{7}{64}$$

$$\frac{57}{64}$$

EXAMPLE 2

1. The probability that a seed will germinate is 0.9. If six seeds are sown what is the probability that
a. five or more seeds will germinate
b. exactly half will germinate

Solution

a.
p of success = 0.9
q of failure = 0.1
number of seeds = 6

$$p(five\ or\ more\ seeds) = p(x = 5) + p(x = 6)$$

$$P(X = x) = nC_x P^x q^{n-x}$$

$x = 5$

$$P(X = 5) = 6C_5 (0.9)^5 (0.1)^{6-5}$$

$$P(X = 5) = 6(0.9)^5 (0.1)^1$$

$$P(X = 5) = 6 \times 0.59049 \times 0.1$$

$$P(X = 5) = 0.354294$$

$x = 6$

$$P(X = 6) = 6C_6 (0.9)^6 (0.1)^{6-6}$$

$$P(X = 6) = 1(0.9)^6 (0.1)^0$$

$$P(X = 6) = 0.531441$$

$$p(five\ or\ more\ seeds) = p(x = 5) + p(x = 6)$$

$$0.354294 + 0.531441$$
$$0.8857.$$

b. exactly half will germinate.

Since there are **6 seeds**, half of them will be **3**. Therefore, the number of seeds expected to germinate is 3

x = 3

$$P(X = 3) = 6C_3 (0.9)^3 (0.1)^{6-3}$$

$$P(X = 3) = 20(0.9)^3 (0.1)^3$$

$$P(X = 3) = 20 \times 0.729 \times 0.001$$

$$P(X = 3) = 0.01458$$

EXAMPLE 3

A farmer produces seeds in packets for sale. The probability that a seed selected will grow is 0.8. If 5 of the seeds are grown what is the probability that

a. less than two will grow

b. less than two will not grow

Solution

$P(X = x) = nC_x\, P^x q^{n-x}$

$P\ success = 0.8$

$q = 1 - 0.8 = 0.2$

$number\ of\ seeds = 5$

$$p(less\ than\ two\ will\ grow) = p(x = 0) + p(x = 1)$$

$x = 0$

$$P(X = 0) = 5C_0\,(0.8)^0(0.2)^{5-0}$$

$$P(X = 0) = 1(0.2)^5$$

$$P(X = 0) = 0.00032$$

$x = 1$

$$P(X = 1) = 5C_1\,(0.8)^1(0.2)^{5-1}$$

$$P(X = 0) = 1(0.2)^5$$

$$P(X = 0) = 0.00032$$

$$p(x = 0) + p(x = 1)$$

0.00032 + 0.00032

0.00064

EXAMPLE 4

A machine produces items; 5% are defective. A sample of 12 items is taken. Find the probability that:

- **(a)** None is defective
- **(b)** At most 1 is defective
- **(c)** At least 2 are defective

Solution:

p (defective) = 0.05
q = 0.95
n = 12

$$P(X = x) = nC_x P^x q^{n-x}$$

(a) P(X = 0):

$$P(X = 0) = 12C_0 \times (0.05)^0 \times (0.95)^{12-0}$$
$$= 1 \times 1 \times (0.95)^{12}$$
$$= 0.5404$$

(b) $\boldsymbol{P(X \leq 1) = P(X = 0) + P(X = 1)}$:

$$P(X = 0) = 12C_0 \times (0.05)^0 \times (0.95)^{12-0}$$
$$= 1 \times 1 \times (0.95)^{12}$$
$$= 0.5404$$

$$P(X = 1) = {}^{12}C1 \times (0.05)1 \times (0.95)12 - 1$$
$$= 12 \times 0.05 \times (0.95)^{11}$$
$$= 12 \times 0.05 \times 0.5688$$
$$= 0.3413$$

$$P(X \leq 1) \approx 0.5404 + 0.3413 = 0.8817$$

(c) $P(X \geq 2) = 1 - P(X \leq 1)$:

$$1 - [P(X = 0) + P(X = 1):]$$
$$= 1 - 0.8817$$
$$= 0.1183$$

EXAMPLE 5

Example 6: For every 10 persons in a city, one is left-handed. Six persons are selected at random. Find the probability that:

- **(a)** Exactly 3 are left-handed
- **(b)** More than half are left-handed
- **(c)** At least 2 are left-handed
- **(d)** None is left-handed

Solution:

p (left-handed) = 1/10 = 0.1
q = 9/10 = 0.9
n = 6

$$P(X = x) = nC_x P^x q^{n-x}$$

(a) P(X = 3):

= $6C_3$ $(0.1)^3(0.9)^{6-3}$
= $6C_3$ $(0.1)^3(0.9)^3$
= 20 × 0.001 × 0.729
= 0.01458

(b) P(X > 3) [more than half of 6 = more than 3]:

= P(X=4) + P(X=5) + P(X=6)

P(X=4) =

= $6C_4$ $(0.1)^4(0.9)^{6-4}$
= 15 $(0.1)^4(0.9)^2$
= 15 × 0.0001 × 0.81
= 0.001215

P(X=5) = $6C_5$ $(0.1)^5(0.9)^{6-5}$

= 6 $(0.1)^5(0.9)^1$
= 6 × 0.00001 × 0.9
= 0.000054

P(X=6) = $6C_6$ $(0.1)^6(0.9)^{6-6}$

= 1 $(0.1)^6(0.9)^0$
= 1× 0.000001 × 1
= 0.000001

P(X>3) = 0.001215 + 0.000054 + 0.000001 = 0.001270

(c) $P(X \geq 2) = 1 - [P(X=0) + P(X=1)]$:
Using complement method to be faster

P(X=0) = $= 6C_0\,(0.1)^0(0.9)^{6-0}$

$= (0.9)^6$

$= 0.531441$

P(X=1) = $= 6C_1\,(0.1)^1(0.9)^{6-1}$

$= 6C_1\,(0.1)^1(0.9)^5$

$= 6 \times (0.1) \times (0.9)^5$

$= 0.354294$

$$P(X \geq 2) = 1 - (0.531441 + 0.354294)$$
$$= \mathbf{1 - 0.885735}$$
$$= \mathbf{0.114265}$$
$$\mathbf{0.1143}$$

(d) P(X=0) = $= 6C_0\,(0.1)^0(0.9)^{6-0}$

$= (0.9)^6$

$= 0.531441$

Practice Questions – Binomial Probability

Note: Attempt all questions. Show full working for full marks.

Q1. A fair coin is tossed 8 times. Find the probability of obtaining:

- **(a)** Exactly 5 heads
- **(b)** At least 6 heads
- **(c)** At most 2 heads
- **(d)** Between 3 and 5 heads inclusive

Answers: (a) 7/32 (b) 37/256 (c) 37/256 (d) 7/16

Q2. The probability that a basketball player scores from the free-throw line is 3/4. In 6 attempts, find the probability that he scores:

- **(a)** All 6
- **(b)** Exactly 4
- **(c)** At least 4
- **(d)** Fewer than 2

Answers: (a) 729/4096 (b) 1080/4096 (c) 3375/4096 ≈ 0.8240 (d) 19/4096

Q3. In a multiple-choice test there are 10 questions, each with 4 options (one correct). A student guesses all answers randomly. Find the probability that the student gets:

- **(a)** Exactly 3 correct
- **(b)** At least 7 correct
- **(c)** A pass of 50% or more

Answers: (a) ${}^{10}C_3(1/4)^3(3/4)^7 \approx 0.2503$ (b) ≈ 0.0035 (c) ≈ 0.0781

Q4. The probability that it rains on any given day in a certain city is 0.3. What is the probability that in a week (7 days):

- **(a)** It rains on exactly 3 days
- **(b)** It rains on at least 5 days
- **(c)** It does not rain at all
- **(d)** It rains on most days (more than half)

Answers: (a) ≈ 0.2269 (b) ≈ 0.0288 (c) ≈ 0.0824 (d) ≈ 0.1260

Q5. Records show that 80% of students who sit for an entrance exam pass. If 10 students sit the exam, find the probability that:

- **(a)** All 10 pass

(b) Exactly 7 pass
(c) At most 6 pass
(d) At least 9 pass

Answers: (a) $(0.8)^{10} \approx 0.1074$ (b) ≈ 0.2013 (c) ≈ 0.1209 (d) ≈ 0.3758

Q6. A marksman has a probability of 0.6 of hitting a target. He fires 7 shots. Calculate the probability that:

(a) He hits exactly 5 times
(b) He misses all 7
(c) He hits at least 3 times
(d) He hits more than 4 times

Answers: (a) ≈ 0.2613 (b) $(0.4)^7 \approx 0.00164$ (c) ≈ 0.7102 (d) ≈ 0.4199

Q7. A survey shows that 1 in 5 households owns a dog. If 9 households are randomly selected, find the probability that:

(a) Exactly 3 own a dog
(b) At most 2 own a dog
(c) At least 4 own a dog

Answers: (a) ≈ 0.1762 (b) ≈ 0.7382 (c) ≈ 0.0856

Q8. A biased coin has P(Head) = 2/3. The coin is tossed 5 times. Find:

(a) P(exactly 4 heads)
(b) P(at least 3 heads)
(c) P(no heads)

Answers: (a) 80/243 (b) 192/243 ≈ 0.7901 (c) 1/243 ≈ 0.0041