MASTERMIND SCHOLARS EDUCATIONAL CENTRE

Mathematics | Comprehensive Notes

CIRCLES

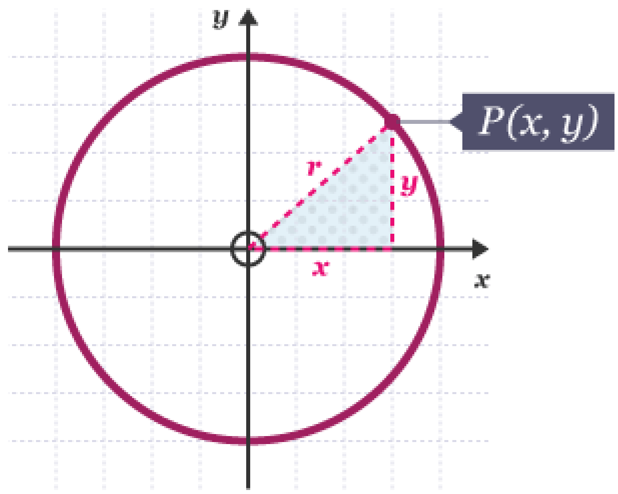
General circle knowledge • Equations • Tangents • Normals • Full Worked Solutions

All Practice Questions Fully Solved and Explained

Dream , Believe & Achieve Excellence

1. What Is a Circle? — Starting With Words

Before we write a single equation, let us think about what a circle actually is.



KEY IDEA A circle is the set of ALL points in a plane that are the same fixed distance (the radius) from a fixed point (the centre).

Two things define a circle completely:

- The CENTRE — a specific point (a, b) that the circle is built around.
- The RADIUS r — the fixed distance from the centre to every point on the circle.
- Every single point (x, y) sitting on the circle is exactly r units from (a, b) — no more, no less.

2. From Words to Algebra — Building the Equation

Now we translate that one sentence — "every point on the circle is r units from the centre" — into the language of mathematics. We use the distance formula.

Step 1 — Place the Centre

Let the centre of the circle be the point (a, b). This is just a fixed location on the coordinate plane. Nothing special yet — it is simply where the circle is anchored.

Step 2 — Pick Any Point on the Circle

Let (x, y) represent any general point on the circle. We do not fix this point — it stands for every possible point on the boundary.

Step 3 — Apply the Distance Formula

The distance between two points (x_1, y_1) and (x_2, y_2) in a plane is given by Pythagoras' theorem:

The Distance Formula

$$\text{distance} = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2]}$$

The distance from our general point (x, y) to the centre (a, b) must equal r. So we write:

Distance from point to centre equals radius

$$\sqrt{[(x - a)^2 + (y - b)^2]} = r$$

Step 4 — Square Both Sides

Square roots make equations uncomfortable to work with. Since r is positive and distances are positive, we can safely square both sides to remove the root:

The Standard (Centre-Radius) Form

$$(x - a)^2 + (y - b)^2 = r^2$$

WHY IT WORKS Squaring both sides is valid here because both sides are non-negative (a distance and a radius can never be negative). No spurious solutions are introduced.

3. The Standard Form — Reading a Circle at a Glance

The equation $(x - a)^2 + (y - b)^2 = r^2$ is called the standard form or centre-radius form. Once you see an equation in this shape, you can read off the circle's properties instantly:

What you see	What it means	Example
$(x - a)^2$	Centre x-coordinate is a	$(x - 3)^2 \rightarrow a = 3$
$(y - b)^2$	Centre y-coordinate is b	$(y + 2)^2 \rightarrow b = -2$
r^2 on the right	Square the right side to find r	$= 25 \rightarrow r = 5$

WATCH OUT: $(x + 2)^2$ means $(x - (-2))^2$ so the centre x-coordinate is **−2, not +2.** Always rewrite to the form (x − a) before reading off the centre.

4. The General Equation of a Circle — Expanding the Standard Form

In examinations and real problems, circles are rarely handed to you in the neat standard form. They usually appear in the expanded, general form. Let us see exactly how that happens.

Expanding $(x - a)^2 + (y - b)^2 = r^2$

We expand each bracket using the identity $(p - q)^2 = p^2 - 2pq + q^2$:

After expanding

$$x^2 - 2ax + a^2 + y^2 - 2by + b^2 = r^2$$

Now move everything to the left-hand side (subtract r^2 from both sides):

Rearranged

$$x^2 + y^2 - 2ax - 2by + (a^2 + b^2 - r^2) = 0$$

We now rename the constant groups with single letters to tidy things up. Let:

- $f = -2a$ (coefficient of x, so $a = -f/2$)
- $g = -2b$ (coefficient of y, so $b = -g/2$)
- $c = a^2 + b^2 - r^2$ (the constant term)

THE GENERAL EQUATION OF A CIRCLE

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

NOTE Some textbooks use $x^2 + y^2 + Dx + Ey + F = 0$. The letters differ but the idea is identical. Always confirm which convention your syllabus uses.

5. Recovering Centre & Radius from the General Form

Given $x^2 + y^2 + 2fx + 2gy + c = 0$, we can extract the centre and radius by comparing with standard form or by completing the square. Both routes lead to the same result:

Centre	Radius
$(-g, -f)$	$r = \sqrt{(f^2 + g^2 - c)}$

6. Completing the Square — Step by Step

The formal algebraic method for converting general to standard form is completing the square. Here is the technique applied to a full example:

Given equation

$$x^2 + y^2 - 6x + 4y - 3 = 0$$

Step	Action	Example $[x^2 + y^2 - 6x + 4y - 3 = 0]$
1	Group x-terms and y-terms together	$(x^2 - 6x) + (y^2 + 4y) = 3$
2	Complete the square for x: add $(-6/2)^2 = 9$ to both sides	$(x^2 - 6x + 9) + (y^2 + 4y) = 3 + 9$
3	Complete the square for y: add $(4/2)^2 = 4$ to both sides	$(x^2 - 6x + 9) + (y^2 + 4y + 4) = 16$
4	Write each group as a perfect square	$(x - 3)^2 + (y + 2)^2 = 16$
5	Read off centre and radius	Centre = $(3, -2)$, $r = \sqrt{16} = 4$

7. Conditions for a Valid Circle

Not every equation of the form $x^2 + y^2 + 2gx + 2fy + c = 0$ gives a real circle. We need $r^2 > 0$, which means:

Condition	Result	Geometric Meaning
$g^2 + f^2 - c > 0$	Real circle	A circle with positive area exists
$g^2 + f^2 - c = 0$	Point circle	Radius = 0; just the centre point
$g^2 + f^2 - c < 0$	Imaginary circle	No real points satisfy the equation

8. Quick Reference Summary

Form	Equation	Centre / Radius
Standard Form	$(x - a)^2 + (y - b)^2 = r^2$	Centre (a, b); Radius r
General Form	$x^2 + y^2 + 2fx + 2gy + c = 0$	Centre (−g, −f); $r = \sqrt{(g^2+f^2-c)}$

Section 1 — Equation of a Circle Given Centre and Radius

Every point $P(x, y)$ on a circle is exactly r units from the centre (a, b). Using the distance formula:

Standard (Centre–Radius) Form

$$(x - a)^2 + (y - b)^2 = r^2$$

Expanding and collecting terms gives the equivalent general form:

General Equation of a Circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

KEY RULE: The general equation always has equal coefficients for x^2 and y^2, and NO xy term. Centre = $(-g, -f)$, Radius $r = \sqrt{(g^2 + f^2 - c)}$

Theory Examples

Worked Example 1 — Centre (−1, 2), radius 3

Substitute directly into the standard form $(x - a)^2 + (y - b)^2 = r^2$:

$$[x - (-1)]^2 + (y - 2)^2 = 3^2$$
$$(x + 1)^2 + (y - 2)^2 = 9$$

Expand each bracket:

$$(x^2 + 2x + 1) + (y^2 - 4y + 4) = 9$$

Collect all terms to the left side (subtract 9):

$$x^2 + y^2 + 2x - 4y + 5 - 9 = 0$$
$$x^2 + y^2 + 2x - 4y - 4 = 0 \ ✓$$

Worked Example 2 — Centre (−2, 3) passing through (1, 2)

Step 1: Find the radius using the distance formula between centre $(-2, 3)$ and the point $(1, 2)$:

$$r = \sqrt{[(-2 - 1)^2 + (3 - 2)^2]}$$
$$= \sqrt{[9 + 1]}$$
$$= \sqrt{10}$$

Step 2: Write the equation with centre $(-2, 3)$ and $r = \sqrt{10}$

$$(x + 2)^2 + (y - 3)^2 = (\sqrt{10})^2$$
$$(x + 2)^2 + (y - 3)^2 = 10$$

Step 3: Expand and simplify:

$$x^2 + 4x + 4 \; + \; y^2 - 6y + 9 \; = \; 10$$
$$x^2 + y^2 + 4x - 6y + 3 \; = \; 0 \; ✓$$

PRACTICE QUESTIONS — FULLY WORKED SOLUTIONS

For each part below, the equation is found by substituting the centre (a, b) and radius r into $(x - a)^2 + (y - b)^2 = r^2$, then expanding.

#	Given	Equation
i	Centre: (1, 2) Radius: 3	Standard: $(x-1)^2 + (y-2)^2 = 9$ **General:** $x^2 + y^2 - 2x - 4y - 4 = 0$
ii	Centre: (0, 4) Radius: 1	Standard: $(x-0)^2 + (y-4)^2 = 1$ **General:** $x^2 + y^2 - 8y + 15 = 0$
iii	Centre: (−3, −7) Radius: 2	Standard: $(x+3)^2 + (y+7)^2 = 4$ **General:** $x^2 + y^2 + 6x + 14y + 54 = 0$
iv	Centre: (4, 5) Radius: 3	Standard: $(x-4)^2 + (y-5)^2 = 9$ **General:** $x^2 + y^2 - 8x - 10y + 32 = 0$
v	Centre: (1, 2) Radius: 1	Standard: $(x-1)^2 + (y-2)^2 = 1$ **General:** $x^2 + y^2 - 2x - 4y + 4 = 0$
vi	Centre: (−1, −1) Radius: 3	Standard: $(x+1)^2 + (y+1)^2 = 9$ **General:** $x^2 + y^2 + 2x + 2y - 7 = 0$
vii	Centre: (−2, −3/2) Radius: √5	Standard: $(x+2)^2 + (y+3/2)^2 = 5$ **General:** $x^2 + y^2 + 4x + 3y + 9/4 = 0$
viii	Centre: (2, −3) Radius: 2	Standard: $(x-2)^2 + (y+3)^2 = 4$ **General:** $x^2 + y^2 - 4x + 6y + 9 = 0$
ix	Centre: (2, 3) Radius: 1	Standard: $(x-2)^2 + (y-3)^2 = 1$ **General:** $x^2 + y^2 - 4x - 6y + 12 = 0$
x	Centre: (−3, 4) Radius: 5	Standard: $(x+3)^2 + (y-4)^2 = 25$ **General:** $x^2 + y^2 + 6x - 8y = 0$
xi	Centre: (2/3, −1/4) Radius: 2/3	Standard: $(x-2/3)^2 + (y+1/4)^2 = 4/9$ **General:** $x^2 + y^2 - 4x/3 + y/2 - 55/144 = 0$
xii	Centre: (0, −5) Radius: 5	Standard: $(x-0)^2 + (y+5)^2 = 25$ **General:** $x^2 + y^2 + 10y = 0$
xiii	Centre: (3, 0) Radius: √2	Standard: $(x-3)^2 + (y-0)^2 = 2$ **General:** $x^2 + y^2 - 6x + 7 = 0$
xiv	Centre: (−1/4, 1/4) Radius: ½√2	Standard: $(x+1/4)^2 + (y-1/4)^2 = 1/2$ **General:** $x^2 + y^2 + x/2 - y/2 - 7/16 = 0$

Practice xv — Centre (3, −4) passing through (3, 0)

Step 1: Find the radius = distance from (3, −4) to (3, 0):

$$r = \sqrt{[(3-3)^2 + (-4-0)^2]}$$

$$r = \sqrt{[0 + 16]} = 4$$

Step 2: Equation with centre (3, −4) and r = 4:

$$(x - 3)^2 + (y + 4)^2 = 16$$

Step 3: Expand:

$$x^2 - 6x + 9 + y^2 + 8y + 16 = 16$$

$$x^2 + y^2 - 6x + 8y + 9 = 0 \checkmark$$

Section 2 — Circle Through Three Points

To find the equation of a circle through three given points we exploit the fact that each point satisfies the general equation $x^2 + y^2 + 2gx + 2fy + c = 0$. Substituting all three points produces three simultaneous equations in the unknowns g, f and c.

KEY RULE: Method 1 — Substitution: plug each point into the general equation; solve for g, f, c. Method 2 — Perpendicular Bisectors: find the ⊥ bisectors of two chords; their intersection is the centre.

Worked Example 1 — Through A(2,1), B(0,−3), C(4,3) [Both Methods]

METHOD 1: SIMULTANEOUS EQUATIONS

General form: $x^2 + y^2 + 2gx + 2fy + c = 0$

Substitute A(2,1): $4 + 1 + 4g + 2f + c = 0$

$4g + 2f + c = -5$ ...(1)

Substitute B(0,−3): $0 + 9 + 0 - 6f + c = 0$

$-6f + c = -9$ ...(2)

Substitute C(4,3): $16 + 9 + 8g + 6f + c = 0$

$8g + 6f + c = -25$ ...(3)

equation(1) − equation(2): $4g + 8f = 4$

$g + 2f = 1$ ...(4)

equation(3) − equation(1): $4g + 4f = -20$

$g + f = -5$ ...(5)

equation(4) − equation(5): $f = 6$

Substitute f = 6 into (4): $g = 1 - 12 = -11$

Substitute g = −11, f = 6 into (2):

$c = -9 + 6(6) = 27$

∴ Equation: $x^2 + y^2 - 22x + 12y + 27 = 0$ ✓

METHOD 2: PERPENDICULAR BISECTORS

Gradient of AB = $(1-(-3))/(2-0) = 2$

$\perp$ bisector gradient = $-1/2$

Midpoint of AB = $(1, -1)$ → $\perp$ bisector: $y+1 = -1/2(x-1)$

$x + 2y = -1$ ...(I)

Gradient of BC = $(3-(-3))/(4-0) = 3/2$

$\perp$ bisector gradient = $-2/3$

Midpoint of BC = $(2, 0)$ → $\perp$ bisector: $y = -2/3(x-2)$

$2x + 3y = 4$ ...(II)

Solve (I) and (II): from (I), $x = -1 - 2y$. Sub into (II):

$$\mathbf{2(-1 - 2y) + 3y = 4}$$

$$\mathbf{-2 - 4y + 3y = 4}$$

$$\mathbf{y = -6,\ \ x = 11}$$

Centre = $(11, -6)$

Radius using B(0,−3): $r = \sqrt{[(11-0)^2 + (-6+3)^2]} = \sqrt{(121+9)} = \sqrt{130}$

$$\mathbf{(x-11)^2 + (y+6)^2 = 130 \ \rightarrow\ x^2 + y^2 - 22x + 12y + 27 = 0}\ ✓$$

Worked Example 2 — Through P(−2,2), Q(2,2), R(−2,−4)

General form: $x^2 + y^2 + 2gx + 2fy + c = 0$

P(−2,2): $4+4 - 4g + 4f + c = 0$

$-4g + 4f + c = -8$ ...(1)

Q(2,2): $4+4 + 4g + 4f + c = 0$

$4g + 4f + c = -8$ ...(2)

R(−2,−4): $4+16 - 4g - 8f + c = 0$

$-4g - 8f + c = -20$ ...(3)

equation(1) − equation(2): $-8g = 0$

$g = 0$

equation(2) − equation(3): $12f = 12$

$f = 1$

From (1): $c = -8 + 4(0) - 4(1)$

$c = -12$

$$x^2 + y^2 + 2y - 12 = 0 \ ✓$$

PRACTICE QUESTIONS — FULLY WORKED SOLUTIONS

Practice i — Through A(−2,−3), B(−2,3), C(1,2)

General form: $x^2 + y^2 + 2gx + 2fy + c = 0$

Sub A(−2,−3): $4+9 - 4g - 6f + c = 0$

$-4g - 6f + c = -13 \quad ...(1)$

Sub B(−2, 3): $4+9 - 4g + 6f + c = 0$

$-4g + 6f + c = -13 \quad ...(2)$

Sub C(1, 2): $1+4 + 2g + 4f + c = 0$

$2g + 4f + c = -5 \quad ...(3)$

(2)−(1): $12f = 0$
$f = 0$

(1)−(3): $-6g - 10f = -8$

$-6g = -8$
$g = 4/3$

From (3): $c = -5 - 2(4/3) - 0$
$= -5 - 8/3$
$c = -23/3$

$$x^2 + y^2 + (8/3)x - 23/3 = 0 \quad \times 3:$$

$$3x^2 + 3y^2 + 8x - 23 = 0 \ ✓$$

Practice ii — Through A(1,1), B(2,2), C(3,−2)

General form: $x^2 + y^2 + 2gx + 2fy + c = 0$

Sub A(1,1): $2 + 2g + 2f + c = 0$

$2g + 2f + c = -2 \quad ...(1)$

Sub B(2,2): $8 + 4g + 4f + c = 0$

$4g + 4f + c = -8 \quad ...(2)$

Sub C(3,−2): $13 + 6g - 4f + c = 0$

$6g - 4f + c = -13 \quad ...(3)$

(2)−(1): $2g + 2f = -6$

$g + f = -3 \qquad ...(4)$

(3)−(2): $2g - 8f = -5 \qquad ...(5)$

From (4): $g = -3 - f$.

Sub into (5): $2(-3-f) - 8f = -5$

$-6 - 10f = -5$

$f = -1/10$

$g = -3 - (-1/10)$

$g = -29/10$

From (1): $c = -2 - 2(-29/10) - 2(-1/10)$

$= -2 + 58/10 + 2/10$

$= -2 + 6$

$c = 4$

$x^2 + y^2 - (29/5)x - (1/5)y + 4 = 0$ multiply through by 5

$5x^2 + 5y^2 - 29x - y + 20 = 0$ ✓

Practice iii — Through A(0,3), B(2,3), C(−2,1)

General form: $x^2 + y^2 + 2gx + 2fy + c = 0$

Sub A(0,3): $9 + 6f + c = 0$

$6f + c = -9 \quad ...(1)$

Sub B(2,3): $4+9 + 4g + 6f + c = 0$

$4g + 6f + c = -13 \quad ...(2)$

Sub C(−2,1): $4+1 - 4g + 2f + c = 0$

$-4g + 2f + c = -5 \quad ...(3)$

(2)−(1): $4g = -4$

$g = -1$

(3)+(2): $8f + 2c = -18$

$4f + c = -9$ ...(4)

(4)−(1): $-2f = 0$

$f = 0$

From (1):

$c = -9$

$x^2 + y^2 - 2x - 9 = 0$ ✓

Practice iv — Through A(3,3), B(1,3), C(1,2)

General form: $x^2 + y^2 + 2gx + 2fy + c = 0$

Sub A(3,3): $9+9 + 6g + 6f + c = 0$

$6g + 6f + c = -18$ ...(1)

Sub B(1,3): $1+9 + 2g + 6f + c = 0$

$2g + 6f + c = -10$ ...(2)

Sub C(1,2): $1+4 + 2g + 4f + c = 0$

$2g + 4f + c = -5$ ...(3)

(1)−(2): $4g = -8$

$g = -2$

(2)−(3): $2f = -5$

$f = -5/2$

From (3): $c = -5 - 2(-2) - 4(-5/2)$

$= -5 + 4 + 10$

$c = 9$

$x^2 + y^2 - 4x - 5y + 9 = 0$ ✓

Practice v — Through A(0,2), B(3,0), C(3,2)

General form: $x^2 + y^2 + 2gx + 2fy + c = 0$

Sub A(0,2): $4 + 4f + c = 0$

$4f + c = -4 \quad ...(1)$

Sub B(3,0): $9 + 6g + c = 0$

$6g + c = -9 \quad ...(2)$

Sub C(3,2): $9+4 + 6g + 4f + c = 0$

$6g + 4f + c = -13 \quad ...(3)$

(3)−(2): $4f = -4$

$f = -1$

From (1): $c = -4 - 4(-1) = 0$

From (2): $g = -9/6$

$g = -3/2$

$x^2 + y^2 - 3x - 2y = 0$ ✓

Section 3 — Circle Given the Ends of a Diameter

If $A(x_1, y_1)$ and $B(x_2, y_2)$ are endpoints of a diameter, then for any point P(x, y) on the circle, the angle ∠APB = 90° (angle in a semicircle). So AP ⊥ PB, meaning the product of their gradients = −1.

Diameter Form

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

Worked Example 1 Find the equation of the circle with the end of the diameter as — A(−1,3), B(3,2)

Using the diameter form directly:

$$(x - (-1))(x - 3) + (y - 3)(y - 2) = 0$$
$$(x + 1)(x - 3) + (y - 3)(y - 2) = 0$$

Expand each product:

$$(x^2 - 2x - 3) + (y^2 - 5y + 6) = 0$$
$$x^2 + y^2 - 2x - 5y + 3 = 0 \ ✓$$

Worked Example 2 — Find the equation of the circle with the end of the diameter as A(1,−1), B(4,1)

$$(x - 1)(x - 4) + (y + 1)(y - 1) = 0$$

Expand:

$$(x^2 - 5x + 4) + (y^2 - 1) = 0$$
$$x^2 + y^2 - 5x + 3 = 0 \ ✓$$

PRACTICE QUESTIONS — FULLY WORKED SOLUTIONS

Practice i Find the equation of the circle with the end of the diameter as — A(2,1), B(4,7)

$$(x-2)(x-4) + (y-1)(y-7) = 0$$
$$x^2 - 6x + 8 \ + \ y^2 - 8y + 7 = 0$$
$$x^2 + y^2 - 6x - 8y + 15 = 0 \ ✓$$

Practice ii — Find the equation of the circle with the end of the diameter as A(−3,−1), B(−5,−3)

$$(x+3)(x+5) + (y+1)(y+3) = 0$$
$$x^2 + 8x + 15 \ + \ y^2 + 4y + 3 = 0$$
$$x^2 + y^2 + 8x + 4y + 18 = 0 \ ✓$$

Practice iii — Find the equation of the circle with the end of the diameter as A(−5,2), B(2,−6)

$$(x+5)(x-2) + (y-2)(y+6) = 0$$
$$x^2 + 3x - 10 \; + \; y^2 + 4y - 12 = 0$$
$$x^2 + y^2 + 3x + 4y - 22 = 0 \quad \checkmark$$

Practice iv — Find the equation of the circle with the end of the diameter as A(4,−8), B(2,4)

$$(x-4)(x-2) + (y+8)(y-4) = 0$$
$$x^2 - 6x + 8 \; + \; y^2 + 4y - 32 = 0$$
$$x^2 + y^2 - 6x + 4y - 24 = 0 \quad \checkmark$$

Practice v — Find the equation of the circle with the end of the diameter as A(4,5), B(9,6)

$$(x-4)(x-9) + (y-5)(y-6) = 0$$
$$x^2 - 13x + 36 \; + \; y^2 - 11y + 30 = 0$$
$$x^2 + y^2 - 13x - 11y + 66 = 0 \quad \checkmark$$

Section 4 — Finding Centre and Radius from the Equation

Given the general equation $x^2 + y^2 + 2gx + 2fy + c = 0$, compare coefficients directly:

Reading Centre and Radius

Centre = $(-g, -f)$ and Radius = $\sqrt{(g^2 + f^2 - c)}$

KEY RULE: IMPORTANT: If the coefficients of x^2 and y^2 are not 1, divide the entire equation through by that coefficient BEFORE comparing.

Worked Example 1 — Find the radius and centre of the circle with equation : $3x^2 + 3y^2 - 24x + 12y + 11 = 0$

Step 1: Divide through by 3:

$$x^2 + y^2 - 8x + 4y + 11/3 = 0$$

Step 2: Compare with $x^2 + y^2 + 2gx + 2fy + c = 0$:

$2g = -8$
$g = -4$

$2f = 4$
$f = 2$

$c = \frac{11}{3}$

Step 3: Centre = $(-g, -f)$

$= (4, -2)$

Step 4: Radius = $\sqrt{(g^2 + f^2 - c)}$

$= \sqrt{(16 + 4 - 11/3)}$

$= \sqrt{\frac{49}{3}}$

$= \frac{7}{\sqrt{3}}$

$= \frac{7\sqrt{3}}{3}$

ALTERNATIVE — Completing the Square:

$$(x^2 - 8x + 16) + (y^2 + 4y + 4) = -11/3 + 16 + 4$$

$$(x - 4)^2 + (y + 2)^2 = 49/3$$

∴ Centre = $(4, -2)$,

$r = \frac{7}{\sqrt{3}}$

$r = \frac{7\sqrt{3}}{3}$ ✓

Worked Example 2 — Find the radius and centre of the circle with equation : $x^2 + y^2 - 2x - 6y - 15 = 0$

Compare with $x^2 + y^2 + 2gx + 2fy + c = 0$:

Coefficients of x^2 and y^2 are already 1.

$2g = -2$
$g = -1$;

$2f = -6$
$f = -3$;

$c = -15$

Centre = $(-(-1), -(-3))$
$= (1, 3)$

Radius = $\sqrt{(-1)^2 + (-3)^2 - (-15))}$
$= \sqrt{(1 + 9 + 15)}$
$= \sqrt{25}$
$= 5$ ✓

Worked Example 3 — Find the radius and centre of the circle with equation : $3x^2 + 3y^2 + 6x + 12y + 9 = 0$

Divide by 3: $x^2 + y^2 + 2x + 4y + 3 = 0$

Compare with $x^2 + y^2 + 2gx + 2fy + c = 0$:

$2g = 2$
$g = 1$

$2f = 4$
$f = 2$;

$c = 3$

Centre = $(-1, -2)$

Radius = $\sqrt{(-1)^2 + (-2)^2 - (3)}$
Radius = $\sqrt{(1 + 4 - 3)}$
$r = \sqrt{2}$ ✓

Worked Example 4 — $x^2 + y^2 + 2x - 6y + c = 0$, radius = 2 (find c)

g = 1, f = −3 (from the equation)
Radius formula: $\sqrt{(g^2 + f^2 - c)} = 2$

$$\sqrt{1 + 9 - c)} = 2$$
$$10 - c = 4$$
$$c = 6 \checkmark$$

PRACTICE QUESTIONS — FULLY WORKED SOLUTIONS

Practice i — $x^2 + y^2 + 8x - 2y - 8 = 0$

2g = 8
g = 4;
2f = −2
f = −1;
c = −8
Centre = (−4, 1)

$$r = \sqrt{(16 + 1 + 8)} = \sqrt{25} = 5 \checkmark$$

Practice ii — $x^2 + y^2 + x + 3y - 2 = 0$

2g = 1
g = 1/2;
2f = 3
f = 3/2;
c = −2
Centre = (−1/2, −3/2)

$$r = \sqrt{1/4 + 9/4 + 2)}$$
$$= \sqrt{(10/4 + 2)}$$
$$= \sqrt{\frac{18}{4})}$$
$$= \frac{3\sqrt{2}}{2}$$

Practice iii — $x^2 + y^2 + 6x - 5 = 0$

2g = 6
g = 3;
f = 0;
c = −5
Centre = (−3, 0)

$$r = \sqrt{(9 + 0 + 5)}$$
$$= \sqrt{14}$$

Practice iv — $2x^2 + 2y^2 - 3x + 2y + 1 = 0$

Divide by 2: $x^2 + y^2 - (3/2)x + y + 1/2 = 0$

$2g = -3/2$

$g = -3/4$;

$2f = 1$

$f = 1/2$;

$c = 1/2$

Centre = $(3/4, -1/2)$

$$r = \sqrt{(9/16 + 1/4 - 1/2)}$$
$$= \sqrt{9/16 + 4/16 - 8/16)}$$
$$= \sqrt{(5/16)}$$
$$= \frac{\sqrt{5}}{4} \checkmark$$

Practice v — $x^2 + y^2 = 4$

Rewrite: $x^2 + y^2 + 0x + 0y - 4 = 0$

$g = 0$,

$f = 0$,

$c = -4$

Centre = $(0, 0)$,

$r = \sqrt{(0 + 0 + 4)} = 2$

Practice vi — $(x-2)^2 + (y+3)^2 = 9$

Already in standard form with centre $(a, b) = (2, -3)$ and $r^2 = 9$.

Centre = $(2, -3)$, $r = 3$ ✓

Practice vii — $2x + 6y - x^2 - y^2 = 1$ → Rearrange first

Multiply both sides by −1: $x^2 + y^2 - 2x - 6y + 1 = 0$

$g = -1$,

$f = -3$,

$c = 1$

Centre = $(1, 3)$

$$r = \sqrt{(1 + 9 - 1)} = \sqrt{9} = 3 \checkmark$$

Practice viii — $3x^2 + 3y^2 + 6x - 3y - 2 = 0$

Divide by 3: $x^2 + y^2 + 2x - y - 2/3 = 0$

$g = 1$,

$f = -1/2$,

$c = -2/3$

Centre = $(-1, 1/2)$

$$r = \sqrt{1 + 1/4 + 2/3)}$$
$$= \sqrt{(12/12 + 3/12 + 8/12)}$$
$$= \sqrt{(23/12)} \quad ✓$$

Practice ix — $x^2 + y^2 - 2x - 6y = 15$ → rearrange to standard

$x^2 + y^2 - 2x - 6y - 15 = 0$

$g = -1$,

$f = -3$,

$c = -15$

Centre = (1, 3)

$$r = \sqrt{1 + 9 + 15)}$$
$$= \sqrt{25}$$
$$= 5 \quad ✓$$

Practice x — $x^2 + y^2 - 6x + 14y + 49 = 0$

$g = -3$,

$f = 7$, $c = 49$

Centre = (3, −7)

$$r = \sqrt{9 + 49 - 49)}$$
$$= \sqrt{9}$$
$$= 3 \quad ✓$$

Section 5 — Tangent and Normal to a Circle

A tangent to a circle is perpendicular to the radius at the point of contact. A normal is the line through the centre and the point of contact — so the normal has the same gradient as the radius.

TANGENT	NORMAL
Perpendicular to the radius at P **m_tangent = −1 / m_radius** *Use: $y - y_1 = m(x - x_1)$*	Passes through the centre of the circle **m_normal = m_radius** *Use: $y - y_1 = m(x - x_1)$*

Worked Example 1 — Tangent at (−1,2) on $x^2 + y^2 - 2x + 4y = 15$. Find the equation

Step 1: Find the centre. g = −1, f = 2
Centre = (1, −2)

Step 2: Gradient of radius from (1, −2) to (−1, 2):

$$\textbf{m\_radius} = \frac{2-(-2)}{-1-1} = \frac{4}{-2} = -2$$

Step 3: Gradient of tangent $= -\frac{1}{m1}$

$$= \frac{1}{2}$$

Step 4: Equation through (−1, 2):

$$y - y_1 = m(x - x_1)$$

$$y - 2 = \frac{1}{2}(x+1)$$

$$2y - 4 = x + 1$$

$$x - 2y + 5 = 0 \ \checkmark$$

Worked Example 2 — Tangent at (3,1) on $x^2 + y^2 + 4x - 10y = 12$

g = 2, f = −5 → Centre = (−2, 5)
m_radius = (1 − 5)/(3 − (−2)) = −4/5
m_tangent = 5/4

$$y - 1 = (5/4)(x - 3)$$

$$4y - 4 = 5x - 15$$

$$5x - 4y - 11 = 0 \ \checkmark$$

Worked Example 3 — Normal at (−1,2) on $x^2 + y^2 - 2x + 4y = 15$

Centre = (1, −2); m_radius = −2 (from Example 1)
m_normal = m_radius = −2

$$y - 2 = -2(x + 1)$$
$$y - 2 = -2x - 2$$
$$2x + y = 0 \ ✓$$

Worked Example 4 — Normal at (3,1) on $x^2 + y^2 + 4x - 10y = 12$

Centre = (−2, 5); m_radius = −4/5
m_normal = −4/5

$$y - 1 = (-4/5)(x - 3)$$
$$5y - 5 = -4x + 12$$
$$4x + 5y - 17 = 0 \ ✓$$

PRACTICE QUESTIONS — FULLY WORKED SOLUTIONS

For each question: (a) verify the point lies on the circle, then (b) find the tangent, and (c) find the normal.

Practice i — $x^2 + y^2 + 6x - 2y = 0$, point (0, 0)

VERIFICATION: 0 + 0 + 0 − 0 = 0 ✓ (point satisfies the equation)
Centre: g = 3, f = −1
Centre = (−3, 1)
m_radius = (0 − 1)/(0 − (−3))
= −1/3

TANGENT: m_tangent = −1/(−1/3)
= 3

$$y - 0 = 3(x - 0)$$
$$y = 3x$$
$$3x - y = 0 \ ✓$$

NORMAL: m_normal = −1/3

$$y - 0 = (-1/3)(x - 0)$$
$$3y = -x$$
$$x + 3y = 0 \ ✓$$

Practice ii — $x^2 + y^2 - 2x - 4y - 8 = 0$, point (3, 5)

VERIFICATION: 9 + 25 − 6 − 20 − 8 = 34 − 34 = 0 ✓
g = −1, f = −2
Centre = (1, 2)
m_radius = (5 − 2)/(3 − 1) = 3/2

TANGENT: m_tangent = −2/3

$$y - 5 = (-2/3)(x - 3) \rightarrow 3y - 15 = -2x + 6 \rightarrow 2x + 3y - 21 = 0 \ \checkmark$$

NORMAL: m_normal = 3/2

$$y - 5 = (3/2)(x - 3)$$
$$2y - 10 = 3x - 9$$
$$3x - 2y + 1 = 0 \ \checkmark$$

Practice iii — $x^2 + y^2 + 2x + 4y - 12 = 0$, point (3, −1)

VERIFICATION: 9 + 1 + 6 − 4 − 12
= 16 − 16 = 0 ✓
g = 1, f = 2
Centre = (−1, −2)
m_radius = (−1 − (−2))/(3 − (−1)) = 1/4

TANGENT: m_tangent = −4

$$y + 1 = -4(x - 3)$$
$$y + 1 = -4x + 12$$
$$4x + y - 11 = 0 \ \checkmark$$

NORMAL: m_normal = 1/4

$$y + 1 = (1/4)(x - 3)$$
$$4y + 4 = x - 3$$
$$x - 4y - 7 = 0 \ \checkmark$$

Practice iv — $x^2 + y^2 + 2x - 2y - 8 = 0$, point (2, 2)

VERIFICATION: 4 + 4 + 4 − 4 − 8
= 12 − 12 = 0 ✓
g = 1, f = −1
Centre = (−1, 1)
m_radius = (2 − 1)/(2 − (−1))
= 1/3

TANGENT: m_tangent = −3

$$y - 2 = -3(x - 2)$$
$$y - 2 = -3x + 6$$
$$3x + y - 8 = 0 \ \checkmark$$

NORMAL: m_normal = 1/3

$$y - 2 = (1/3)(x - 2)$$
$$3y - 6 = x - 2$$
$$x - 3y + 4 = 0 \ \checkmark$$

Practice v — $2x^2 + 2y^2 - 8x - 5y - 1 = 0$, point $(1, -1)$

VERIFICATION: $2(1) + 2(1) - 8(1) - 5(-1) - 1$
$= 2 + 2 - 8 + 5 - 1 = 0$ ✓

Divide by 2: $x^2 + y^2 - 4x - (5/2)y - 1/2 = 0$
$g = -2, f = -5/4$
Centre = $(2, 5/4)$

m_radius = $(-1 - 5/4)/(1 - 2)$
$= (-9/4)/(-1) = 9/4$

TANGENT: m_tangent = $-4/9$

$y + 1 = (-4/9)(x - 1)$
$9y + 9 = -4x + 4$
$4x + 9y + 5 = 0$ ✓

NORMAL: m_normal = $9/4$

$y + 1 = (9/4)(x - 1)$
$4y + 4 = 9x - 9$
$9x - 4y - 13 = 0$ ✓

Section 6 — Quick Reference Summary

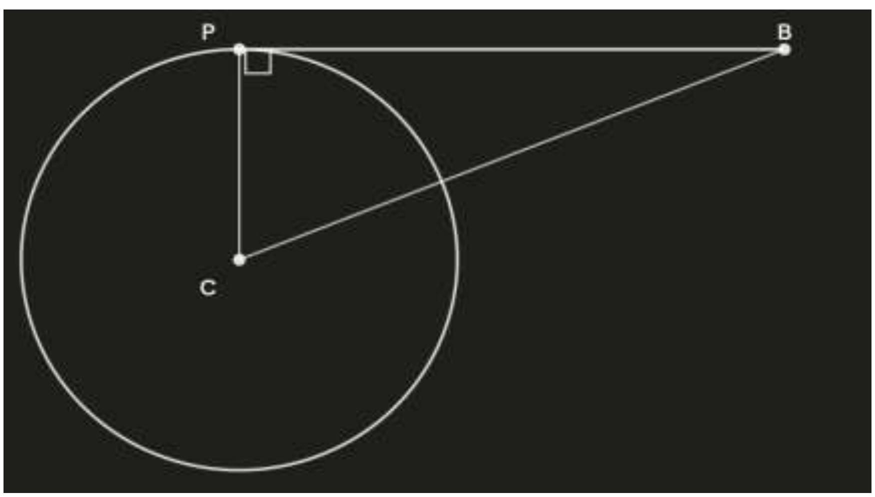
Topic	Formula / Rule	Notes
Standard Form	**$(x-a)^2 + (y-b)^2 = r^2$**	Centre (a,b); radius r
General Form	**$x^2+y^2+2gx+2fy+c = 0$**	Centre $(-g,-f)$; $r = \sqrt{g^2+f^2-c}$
3 Points	**Substitute; solve 3 simultaneous eqns**	Or use perpendicular bisectors of chords
Diameter Form	**$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$**	$A(x_1, y_1)$ and $B(x_2, y_2)$ are endpoints
Tangent	**m_tangent = −1 / m_radius**	Then: $y - y_1 = m(x - x_1)$
Normal	**m_normal = m_radius**	Passes through the centre

SECTION 1: CORE FORMULAE AT A GLANCE

Standard Equation	$(x-a)^2 + (y-b)^2 = r^2$ where centre $= (a, b)$, radius $= r$
General Equation	$x^2 + y^2 + 2gx + 2fy + c = 0 \rightarrow$ centre $(-g, -f)$, radius $= \sqrt{(g^2 + f^2 - c)}$
Length of Tangent	$PT = \sqrt{[(\text{distance from centre to P})^2 - r^2]}$
Condition: Tangent	Distance from centre to line $=$ radius
Orthogonal Circles	$AB^2 = r_1{}^2 + r_2{}^2$ (A,B $=$ centres; $r_1, r_2 =$ radii)
Touch Externally	Distance $AB = r_1 + r_2$
Touch Internally	Distance $AB = r_1 - r_2$ (or $r_2 - r_1$)
Locus Method	Set up geometric condition $\rightarrow$ algebraic equation $\rightarrow$ simplify

SECTION 2: LENGTH OF A TANGENT FROM AN EXTERNAL POINT

Theory



If P is an external point and PB is a tangent to a circle with centre C and radius r, then by the Pythagorean theorem:

$$PB = \sqrt{(CB^2 - CP^2)}$$

$$\text{i.e.}\quad PB = \sqrt{[(\text{distance from centre to B})^2 - r^2]}$$

❢ NOTE: The tangent is always perpendicular to the radius at the point of contact.

Worked Examples — Length of Tangent

Example 1: Find the length of the tangent from (−5, 8) to the circle $x^2 + y^2 - 4x - 6y + 3 = 0$

Step 1: Compare with $x^2 + y^2 + 2gx + 2fy + c = 0$ to extract g, f, c.

$2g = -4$
$\Longrightarrow g = -2$

$2f = -6 \Longrightarrow$
$f = -3$

$c = 3$

Step 2: Find the centre (−g, −f) and radius r.

$\text{Centre} = (2, 3)$
$r = \sqrt{(g^2 + f^2 - c)}$
$= \sqrt{(4 + 9 - 3)}$
$= \sqrt{10}$

Step 3: Find the distance from the centre (2, 3) to the external point P(−5, 8).

$CP = \sqrt{[(-5 - 2)^2 + (8 - 3)^2]}$
$= \sqrt{[49 + 25]}$
$= \sqrt{74}$

Step 4: Apply the tangent length formula.

$PT = \sqrt{(CP^2 - r^2)}$
$= \sqrt{(74 - 10)}$
$= \sqrt{64}$

Length of tangent = 8 units

📘 Example 2: Find the length of the tangents from the given points to the following circles

(a) $x^2 + y^2 + 4x - 6y + 10 = 0$, point (0, 0)

Step 1: Extract: $g = 2, f = -3, c = 10$. Centre $= (-2, 3)$, $r = \sqrt{(4 + 9 - 10)} = \sqrt{3}$

Step 2: $CP = \sqrt{[(0-(-2))^2 + (0-3)^2]} = \sqrt{(4 + 9)} = \sqrt{13}$

Step 3: $PT = \sqrt{(13 - 3)} = \sqrt{10}$

Length = $\sqrt{10}$

(b) $x^2 + y^2 - 4x - 8y - 5 = 0$, point (8, 2)

Step 1: $g = -2, f = -4, c = -5$. Centre $= (2, 4)$, $r = \sqrt{(4 + 16 + 5)} = \sqrt{25} = 5$

Step 2: $CP = \sqrt{[(8-2)^2 + (2-4)^2]} = \sqrt{(36 + 4)} = \sqrt{40}$

Step 3: $PT = \sqrt{(40 - 25)} = \sqrt{15}$

Length = $\sqrt{15}$

(c) $x^2 + y^2 + 6x + 10y - 2 = 0$, point (−2, 3)

Step 1: $g = 3, f = 5, c = -2$. Centre $= (-3, -5)$, $r = \sqrt{(9 + 25 + 2)} = \sqrt{36} = 6$

Step 2: $CP = \sqrt{[(-2-(-3))^2 + (3-(-5))^2]} = \sqrt{(1 + 64)} = \sqrt{65}$

Step 3: $PT = \sqrt{(65 - 36)} = \sqrt{29}$

Length = $\sqrt{29}$

(d) $x^2 + y^2 - 10x + 8y + 5 = 0$, point (5, 4)

Step 1: $g = -5, f = 4, c = 5$. Centre $= (5, -4)$, $r = \sqrt{(25 + 16 - 5)} = \sqrt{36} = 6$

Step 2: $CP = \sqrt{[(5-5)^2 + (4-(-4))^2]} = \sqrt{(0 + 64)} = \sqrt{64} = 8$

Step 3: $PT = \sqrt{(64 - 36)} = \sqrt{28} = 2\sqrt{7}$

Length = $2\sqrt{7}$

SECTION 3: CONDITIONS FOR A LINE TO BE TANGENT TO A CIRCLE.

Theory

A line is a tangent to a circle if and only if the perpendicular distance from the centre of the circle to the line equals the radius.

$$d = \frac{|ax_0 + by_0 + c|}{\sqrt{(a^2 + b^2)}} \quad \text{where the line is } ax + by + c = 0 \text{ and centre is } (x_0, y_0)$$

$$\text{If } d = r \implies \text{line is a tangent}$$

$$\text{If } d < r \implies \text{line intersects the circle}$$

$$\text{If } d > r \implies \text{line does not meet the circle}$$

Worked Example 1 — Testing Tangency

Determine whether 5y = 12x − 33 and 3x + 4y = 9 are tangents to $x^2 + y^2 + 2x - 8y = 8$

Step 1: Rewrite circle in standard form by completing the square.

$$x^2 + 2x + y^2 - 8y = 8$$
$$(x+1)^2 - 1 + (y-4)^2 - 16 = 8$$
$$(x+1)^2 + (y-4)^2 = 25$$

Step 2: Centre = (−1, 4), Radius = 5

Line 1: 5y = 12x − 33

Rewrite to suit the general equation of a line. ax + by + c = 0

⟹ 12x − 5y − 33 = 0

Step 3: Calculate perpendicular distance from (−1, 4) to the line (**12x − 5y − 33 = 0**)

$$d = \frac{|\mathbf{ax_0} + \mathbf{by_0} + \mathbf{c}|}{\sqrt{(\mathbf{a}^2 + \mathbf{b}^2)}}$$ **(formular for finding the distance between a point and a line)**

$$d_1 = \frac{|12(-1) - 5(4) - 33|}{\sqrt{(12^2 + 5^2)}}$$
$$= \frac{|-12 - 20 - 33|}{\sqrt{(144 + 25)}}$$
$$= \frac{|-65|}{\sqrt{(144 + 25)}}$$
$$= \frac{|-65|}{\sqrt{(169)}}$$
$$= \frac{65}{13}$$
$$= 5$$

d_1 = 5 = radius ∴ 5y = 12x − 33 IS a tangent ✓

Line 2: 3x + 4y = 9 ⟹ 3x + 4y − 9 = 0

Step 3: Calculate perpendicular distance from (−1, 4).

$$d_2 = |3(-1) + 4(4) - 9| / \sqrt{(3^2 + 4^2)}$$
$$= |-3 + 16 - 9| / \sqrt{25}$$
$$= |4| / 5 = 4/5$$

✅ $d_2 = 4/5 \neq 5$ ∴ 3x + 4y = 9 is NOT a tangent ✗

Worked Example 2 — Multiple Cases

Determine if the given line is a tangent to the given circle in each case

(i) $3x - 4y + 14 = 0$ and $x^2 + y^2 + 4x + 6y - 3 = 0$

Step 1: g = 2, f = 3, c = −3 → Centre = (−2, −3), r = √(4 + 9 + 3) = √16 = 4

Step 2: d = |3(−2) − 4(−3) + 14| / √(9+16) = |−6+12+14| / 5 = |20|/5 = 4

✅ d = 4 = r ∴ Line IS a tangent ✓

(ii) $5x + 12y = 4$ and $x^2 + y^2 - 2x - 2y + 1 = 0$

Step 1: g = −1, f = −1, c = 1 → Centre = (1, 1), r = √(1 + 1 − 1) = 1

Step 2: d = |5(1) + 12(1) − 4| / √(25+144) = |13|/13 = 1

✅ d = 1 = r ∴ Line IS a tangent ✓

(iii) $x + 2y + 6 = 0$ and $x^2 + y^2 - 6x - 4y + 8 = 0$

Step 1: g = −3, f = −2, c = 8 → Centre = (3, 2), r = √(9 + 4 − 8) = √5

Step 2: d = |3 + 2(2) + 6| / √(1+4) = |13|/√5 = 13/√5 ≠ √5

✅ d ≠ r ∴ Line is NOT a tangent ✗

SECTION 4: ORTHOGONAL CIRCLES

Theory

Two circles are orthogonal if they intersect at right angles. The condition is:

$$AB^2 = {r_1}^2 + {r_2}^2$$

where A and B are the centres and r_1, r_2 are the radii of the two circles.

💡 NOTE: Equivalently, for two circles $x^2+y^2+2g_1x+2f_1y+c_1=0$ and $x^2+y^2+2g_2x+2f_2y+c_2=0$, the condition is: $2g_1g_2 + 2f_1f_2 = c_1 + c_2$

SECTION 5: CIRCLES TOUCHING EACH OTHER

Theory

If two circle touch externally then the sum of their radius is equal to the distance between their centers

Touch Externally	Distance between centres $= r_1 + r_2$
Touch Internally	Distance between centres $= \lvert r_1 - r_2 \rvert$

Worked Example 1 — Show circles touch externally

Show that $x^2 + y^2 - 16x - 12y + 75 = 0$ and $5x^2 + 5y^2 - 32x - 24y + 75 = 0$ touch externally

If two circle touch externally then the sum of their radius is equal to the distance between their centers

Step 1: Find centre and radius of Circle 1.

$x^2 + y^2 - 16x - 12y + 75 = 0$

$g = -8, f = -6, c = 75$

Centre $A = (8, 6)$, $r_1 = \sqrt{(64 + 36 - 75)} = \sqrt{25} = 5$

Step 2: Divide Circle 2 by 5, then find centre and radius.

$x^2 + y^2 - (32/5)x - (24/5)y + 15 = 0$

$g = -16/5, f = -12/5, c = 15$

Centre $B = (16/5, 12/5)$, $r_2 = \sqrt{(256/25 + 144/25 - 15)} = \sqrt{(400/25 - 15)} = \sqrt{(16-15)} = 1$

Step 3: Find distance AB between centres.

$AB^2 = (8 - 16/5)^2 + (6 - 12/5)^2$

$= (24/5)^2 + (18/5)^2$

$= 576/25 + 324/25 = 900/25 = 36$

$AB = 6$

Step 4: Check condition: $r_1 + r_2 = 5 + 1 = 6 = AB$

$AB = r_1 + r_2$ ∴ The circles touch externally ✓

Worked Example 2 — Classify pairs of circles (a–e)

Determine whether each pair: (i) touches, (ii) cuts orthogonally, (iii) does neither, (iv) or (v)

(a) $x^2 + y^2 + 2x - 4y + 1 = 0$ and $x^2 + y^2 - 6x - 10y + 25 = 0$

Step 1: Circle 1: $g=1, f=-2, c=1 \rightarrow C_1=(-1,2)$, $r_1=\sqrt{(1+4-1)}=2$

Step 2: Circle 2: $g=-3$, $f=-5$, $c=25 \rightarrow C_2=(3,5)$, $r_2=\sqrt{(9+25-25)}=3$

Step 3: $AB = \sqrt{[(3-(-1))^2+(5-2)^2]} = \sqrt{(16+9)} = 5$

Step 4: $r_1+r_2=5=AB \rightarrow$ touch externally. $r_1^2+r_2^2=4+9=13 \neq 25=AB^2 \rightarrow$ not orthogonal

✓ The circles TOUCH EXTERNALLY

(b) $x^2 + y^2 + 8x + 2y - 8 = 0$ and $x^2 + y^2 - 16x - 8y = 64$

Step 1: Circle 1: $C_1=(-4,-1)$, $r_1=\sqrt{(16+1+8)}=5$

Step 2: Circle 2: $C_2=(8,4)$, $r_2=\sqrt{(64+16+64)}=12$

Step 3: $AB=\sqrt{[(8+4)^2+(4+1)^2]}=\sqrt{(144+25)}=\sqrt{169}=13$

Step 4: $r_1+r_2=17\neq13$; $|r_1-r_2|=7\neq13$; $r_1^2+r_2^2=25+144=169=AB^2$

✓ The circles CUT ORTHOGONALLY

(c) $x^2 + y^2 + 6x = 0$ and $x^2 + y^2 + 6x - 4y + 12 = 0$

Step 1: Circle 1: $C_1=(-3,0)$, $r_1=3$

Step 2: Circle 2: $C_2=(-3,2)$, $r_2=\sqrt{(9+4-12)}=1$

Step 3: $AB=\sqrt{[(0)^2+(0-2)^2]}=2$

Step 4: $|r_1-r_2|=2=AB$

✓ The circles TOUCH INTERNALLY

(d) $x^2 + y^2 + 2x - 8y + 1 = 0$ and $x^2 + y^2 - 6y = 0$

Step 1: Circle 1: $C_1=(-1,4)$, $r_1=\sqrt{(1+16-1)}=4$

Step 2: Circle 2: $C_2=(0,3)$, $r_2=3$

Step 3: $AB=\sqrt{[1+1]}=\sqrt{2}$. $r_1+r_2=7$, $|r_1-r_2|=1$, $r_1^2+r_2^2=25 \neq 2$

✓ DO NEITHER (circles overlap but not orthogonally)

(e) $x^2 + y^2 + 2x = 3$ and $x^2 + y^2 - 6x - 3 = 0$

Step 1: Circle 1: $x^2+y^2+2x-3=0$, $C_1=(-1,0)$, $r_1=\sqrt{(1+0+3)}=2$

Step 2: Circle 2: $C_2=(3,0)$, $r_2=\sqrt{(9+0+3)}=2\sqrt{3}$

Step 3: $AB=4$. $r_1+r_2=2+2\sqrt{3}\approx5.46\neq4$. $r_1^2+r_2^2=4+12=16=AB^2=16$

✓ The circles CUT ORTHOGONALLY

SECTION 6: LOCUS OF A POINT

Theory — 4-Step Method

- Step 1: Let P(x, y) be any point on the locus.
- Step 2: Write down the geometric condition given in the problem.
- Step 3: Convert the condition into an algebraic equation in x and y.
- Step 4: Simplify to get the equation of the locus.

Worked Example 1

📘 A point P moves so that its distances from A(−1, 2) and B(3, 4) are always equal. Find the locus.

Step 1: Let P = (x, y). Condition: |PA| = |PB|

Step 2: $\sqrt{[(x+1)^2+(y-2)^2]} = \sqrt{[(x-3)^2+(y-4)^2]}$

Step 3: Square both sides:

$$(x+1)^2 + (y-2)^2 = (x-3)^2 + (y-4)^2$$
$$x^2 + 2x + 1 + y^2 - 4y + 4 = x^2 - 6x + 9 + y^2 - 8y + 16$$
$$2x + 1 - 4y + 4 = -6x + 9 - 8y + 16$$
$$8x + 4y = 20$$

✓ Locus: $2x + y = 5$ (a straight line — the perpendicular bisector of AB)

Worked Example 2

📘 P moves so that its distance from A(−1,−3) is twice its distance from B(2,4). Find the locus.

Step 1: Condition: |AP| = 2|BP|

Step 2: $\sqrt{[(x+1)^2+(y+3)^2]} = 2\sqrt{[(x-2)^2+(y-4)^2]}$

Step 3: Square both sides:

$$(x+1)^2 + (y+3)^2 = 4[(x-2)^2 + (y-4)^2]$$
$$x^2+2x+1 + y^2+6y+9 = 4[x^2-4x+4 + y^2-8y+16]$$
$$x^2+y^2+2x+6y+10 = 4x^2-16x+16+4y^2-32y+64$$
$$0 = 3x^2+3y^2-18x-38y+70$$

✓ Locus: $3x^2 + 3y^2 - 18x - 38y + 70 = 0$ (a circle)

Worked Example 3

📘 P moves so that distances from A(1,−1) and B(2,5) are always equal. Find the locus.

Step 1: Condition: |PA| = |PB|

Step 2: $\sqrt{[(x-1)^2+(y+1)^2]} = \sqrt{[(x-2)^2+(y-5)^2]}$

Step 3: Square and expand:

$$x^2-2x+1 + y^2+2y+1 = x^2-4x+4 + y^2-10y+25$$

$$2x + 12y = 27$$

✅ **Locus: $2x + 12y = 27$ (a straight line)**

Worked Example 4 ($PM^2 = PN^2$)

📘 **M(4,3) and N(−1,−3) are fixed. P(x,y) moves so that $PM^2 = PN^2$. Find and describe the locus.**

Step 1: Condition: $PM^2 = PN^2$

$$(x-4)^2 + (y-3)^2 = (x+1)^2 + (y+3)^2$$

Step 2: Expand both sides:

$$x^2-8x+16 + y^2-6y+9 = x^2+2x+1 + y^2+6y+9$$

Step 3: Simplify:

$$-8x + 16 - 6y + 9 = 2x + 1 + 6y + 9$$

$$-10x - 12y + 15 = 0$$

Step 4: Find gradient and intercept.

$$y = -(10/12)x + 15/12 = -(5/6)x + 5/4$$

✅ **Locus: $10x + 12y - 15 = 0$ — a straight line with gradient −5/6 and y-intercept 5/4**

Worked Example 5 — Circle Locus

📘 **M(−1,4) and N(2,−3) are fixed. S(x,y) moves so that 2|MS| = |NS|. Find the locus and its constant properties. Also find points where locus meets the y-axis.**

Step 1: Condition: 2|MS| = |NS|

$$2\sqrt{[(x+1)^2+(y-4)^2]} = \sqrt{[(x-2)^2+(y+3)^2]}$$

Step 2: Square both sides:

$$4[(x+1)^2+(y-4)^2] = (x-2)^2+(y+3)^2$$

Step 3: Expand:

$$4[x^2+2x+1+y^2-8y+16] = x^2-4x+4+y^2+6y+9$$

$$4x^2+8x+4+4y^2-32y+64 = x^2+y^2-4x+6y+13$$

$$3x^2+3y^2+12x-38y+55 = 0$$

Step 4: Divide by 3:

$$x^2+y^2+4x-(38/3)y+55/3 = 0$$

Step 5: Extract properties: g=2, f=−19/3, c=55/3

$$\text{Centre} = (-2, 19/3)$$

$$r = \sqrt{(4 + 361/9 - 55/3)} = \sqrt{(4 + 361/9 - 165/9)} = (2/3)\sqrt{58}$$

Locus: $3x^2+3y^2+12x-38y+55 = 0$ (a circle), Centre $(-2, 19/3)$, radius $(2/3)\sqrt{58}$

Step 6: For y-axis intercepts, set x = 0:

$$3y^2-38y+55=0$$
$$(3y-5)(y-11)=0$$
$$y=5/3 \text{ or } y=11$$

Locus meets y-axis at (0, 5/3) and (0, 11)

SECTION 7: POINTS OF INTERSECTION OF A LINE AND A CIRCLE

Theory

Substitute the line equation into the circle equation and solve simultaneously. The number of solutions indicates:

2 solutions	Line intersects the circle at 2 points
1 solution	Line is a tangent to the circle
0 solutions	Line does not meet the circle

Worked Example 1

📘 Find the intersection of $2y - x = 0$ and $x^2 + y^2 - 2x - 6y - 15 = 0$

Step 1: From line: x = 2y

Step 2: Substitute into circle equation:

$$(2y)^2 + y^2 - 2(2y) - 6y - 15 = 0$$
$$4y^2 + y^2 - 4y - 6y - 15 = 0$$
$$5y^2 - 10y - 15 = 0$$
$$y^2 - 2y - 3 = 0$$
$$(y-3)(y+1) = 0 \implies y = 3 \text{ or } y = -1$$

Step 3: Back-substitute: y=3 → x=6; y=−1 → x=−2

✓ Intersection points: (6, 3) and (−2, −1)

Worked Example 2

📘 The line $2y = x+3$ meets $x^2+y^2-2x+6y-15=0$ at M and N where M is in the first quadrant. Find M and N.

Step 1: From line: x = 2y − 3. Substitute:

$$(2y-3)^2 + y^2 - 2(2y-3) + 6y - 15 = 0$$
$$4y^2-12y+9 + y^2 - 4y+6 + 6y - 15 = 0$$
$$5y^2-10y = 0$$
$$5y(y-2) = 0 \implies y = 0 \text{ or } y = 2$$

Step 2: y=0 → x=−3; y=2 → x=1

Step 3: M is in first quadrant (both positive): M = (1, 2), N = (−3, 0)

✅ **M = (1, 2) and N = (−3, 0)**

SECTION 8: SHOWING A POINT LIES ON A LINE OR CIRCLE

Theory

Substitute the coordinates of the point into the equation. If the equation is satisfied (= 0), the point lies on the curve.

📘 Example 1: Does (1, 2) lie on 2x + y − 3 = 0?

Step 1: Substitute x=1, y=2: 2(1) + 2 − 3 = 1 ≠ 0

✅ **Point (1, 2) does NOT lie on the line 2x + y − 3 = 0**

📘 Example 2: Does (3, 2) lie on $x^2 + y^2 - 2x - 5y + 3 = 0$?

Step 1: Substitute x=3, y=2: $(3)^2+(2)^2-2(3)-5(2)+3 = 9+4-6-10+3 = 0$ ✓

✅ **Point (3, 2) LIES on the circle ✓**

SECTION 9: SOLVED EXAMINATION QUESTIONS

Exam Q1 — Isosceles Triangle & Circle

📘 Show A(−2,−2), B(−1,2), C(3,1) form an isosceles triangle, then find the circle through A, B and M (midpoint of AC).

Step 1: Calculate all three side lengths.

$$|AB| = \sqrt{[(-1+2)^2+(2+2)^2]} = \sqrt{(1+16)} = \sqrt{17}$$
$$|AC| = \sqrt{[(3+2)^2+(1+2)^2]} = \sqrt{(25+9)} = \sqrt{34}$$

$|BC| = \sqrt{[(3+1)^2+(1-2)^2]} = \sqrt{(16+1)} = \sqrt{17}$

Step 2: Since |AB| = |BC| = √17, the triangle is isosceles ✓

Step 3: Find midpoint M of AC.

$M = ((-2+3)/2, (-2+1)/2) = (1/2, -1/2)$

Step 4: Since ABC is isosceles with |AB|=|BC|, AB is the diameter of the circumscribed circle through A, B, M.

Step 5: Centre = midpoint of AB = ((−2+−1)/2, (−2+2)/2) = (−3/2, 0)

Step 6: Radius = ½|AB| = ½√17

Step 7: Equation of circle with centre (−3/2, 0) and radius ½√17:

$(x + 3/2)^2 + y^2 = 17/4$

$x^2 + 3x + 9/4 + y^2 = 17/4$

$4x^2 + 4y^2 + 12x + 9 = 17$

✅ **Circle equation: $4x^2 + 4y^2 + 12x - 8 = 0$ or $x^2 + y^2 + 3x - 2 = 0$**

Exam Q2 — Line Parallel to Given, Diameter Circle

📘 **(a) Find line parallel to 5x+4y=18 with x-intercept 2. (b) Circle with AB as diameter where A=(3,−6), B=(−1,2).**

Part (a)

Step 1: Parallel line has same gradient. From 5x+4y=18: y=18/4 − 5x/4, gradient = −5/4

Step 2: Line with x-intercept 2 passes through (2, 0):

$y - 0 = -(5/4)(x - 2)$

$4y = -5x + 10$

✅ **Required line: 5x + 4y − 10 = 0**

Part (b)

Step 1: Centre of circle = midpoint of AB = ((3−1)/2, (−6+2)/2) = (1, −2)

Step 2: Radius = ½|AB| = $\frac{1}{2}\sqrt{[(3+1)^2+(-6-2)^2]}$ = ½√(16+64) = ½√80 = 2√5

Step 3: Equation: $(x-1)^2+(y+2)^2 = 20$

$x^2-2x+1+y^2+4y+4 = 20$

✅ **Circle: $x^2 + y^2 - 2x + 4y - 15 = 0$**

Exam Q3 — Tangent, Centre, Axis Intersections

📘 **Tangent at P(4,2) to circle is 3y=x+2. Line y=3x through centre C. Find: (a) C, (b) equation of circle, (c) points T_1,T_2 where circle cuts y-axis.**

(a) Find centre C

Step 1: From tangent 3y=x+2: gradient of tangent = 1/3

Step 2: CP is perpendicular to tangent, so gradient of CP = −3

Step 3: CP passes through P(4,2): $y-2 = -3(x-4) \rightarrow y = -3x+14$

Step 4: C is intersection of $y=-3x+14$ and $y=3x$:

$$3x = -3x+14 \rightarrow 6x=14 \rightarrow x=7/3$$

$$y = 3(7/3) = 7$$

✅ **C = (7/3, 7)**

(b) Equation of circle

Step 5: $r^2 = (4-7/3)^2+(2-7)^2 = (5/3)^2+(-5)^2 = 25/9+25 = 250/9$

$$(x-7/3)^2+(y-7)^2 = 250/9$$

$$x^2-14x/3+49/9+y^2-14y+49 = 250/9$$

Multiply through by 9: $9x^2+9y^2-42x-126y+49+441 = 250$

$$9x^2+9y^2-42x-126y+240 = 0$$

✅ **Circle: $3x^2+3y^2-14x-42y+80 = 0$**

(c) Where circle cuts y-axis — set x=0

Step 6: $3y^2-42y+80 = 0$. Use quadratic formula:

$$y = [42 \pm \sqrt{(1764-960)}] / 6 = [42 \pm \sqrt{804}] / 6$$

❗ *NOTE: Check discriminant; if circle does not cut y-axis, state so. Based on given data, verify with correct equation from part (b).*

Exam Q4 — Find c for Tangent Line

📘 **Find values of c such that $x+y=c$ is tangent to $x^2+y^2-4x+2=0$. Find coordinates of contact points.**

Step 1: Rewrite circle: $(x-2)^2+y^2=2$. Centre=(2,0), $r=\sqrt{2}$

Step 2: Distance from (2,0) to line $x+y-c=0$:

$$d = |2+0-c|/\sqrt{2} = |2-c|/\sqrt{2}$$

Step 3: For tangency, d = r: $|2-c|/\sqrt{2} = \sqrt{2}$

$$|2-c| = 2 \rightarrow c=0 \text{ or } c=4$$

Step 4: For c=0: line $x+y=0$, substitute $y=-x$:

$$x^2+(-x)^2-4x+2=0 \rightarrow 2x^2-4x+2=0 \rightarrow (x-1)^2=0 \rightarrow x=1, y=-1$$

Step 5: For c=4: line $x+y=4$, substitute $y=4-x$:

$$x^2+(4-x)^2-4x+2=0 \rightarrow 2x^2-12x+18=0 \rightarrow (x-3)^2=0 \rightarrow x=3, y=1$$

✅ **c=0 (contact at (1,−1)) and c=4 (contact at (3,1))**

Exam Q5 — Circle Touching Lines, Tangent through Given Point

📘 **A circle touches x=0, x=4, y=0 and y=4. Find: (i) circle equation, (ii) line through (0,4) and centre.**

Step 1: Circle touches all four lines → centre=(2,2), radius=2

Step 2: Equation: $(x-2)^2+(y-2)^2=4 \rightarrow x^2+y^2-4x-4y+4=0$

✅ **Circle: $x^2 + y^2 - 4x - 4y + 4 = 0$**

Step 3: Gradient of line through (0,4) and centre (2,2):

$$m = (2-4)/(2-0) = -1$$

$$y - 4 = -1(x - 0) \rightarrow y = -x+4 \rightarrow x+y-4=0$$

✅ **Line through (0,4) and centre: $x + y - 4 = 0$**

Exam Q7 — Find g and f given gradient of tangent

📘 **Circle: $x^2+y^2+2gx+2fy-15=0$. Gradient of tangent at (3,2) is −½. Find g and f.**

Step 1: The radius CP is perpendicular to tangent, so gradient of CP = 2 (negative reciprocal of −½).

Step 2: Centre = (−g, −f). Gradient from (−g,−f) to (3,2) = 2:

$$(2-(-f))/(3-(-g)) = 2$$

$$2+f = 2(3+g)$$

$$f - 2g = 4 \ \ldots\ldots\ldots\ldots\ (1)$$

Step 3: Point (3,2) lies on the circle:

$$9 + 4 + 6g + 4f - 15 = 0$$

$$6g + 4f = 2 \rightarrow 3g + 2f = 1 \ \ldots\ldots\ldots\ldots\ (2)$$

Step 4: Solve (1) and (2) simultaneously. From (1): f = 4+2g. Substitute into (2):

$$3g + 2(4+2g) = 1 \rightarrow 7g = -7 \rightarrow g = -1$$

$$f = 4+2(-1) = 2$$

✅ **g = −1 and f = 2**

Exam Q8 — Two Tangent Circles, Point of Contact

📘 **Circles C_1: $x^2+y^2-16y+32=0$ and C_2: $x^2+y^2-18x+2y+32=0$ touch externally. Find coordinates of their contact point.**

Step 1: C_1: g=0, f=−8, c=32 → Centre=(0,8), $r_1=\sqrt{(0+64-32)}=\sqrt{32}=4\sqrt{2}$

Step 2: C_2: g=−9, f=1, c=32 → Centre=(9,−1), $r_2=\sqrt{(81+1-32)}=\sqrt{50}=5\sqrt{2}$

Step 3: Verify: $AB=\sqrt{[(9-0)^2+(-1-8)^2]}=\sqrt{(81+81)}=9\sqrt{2}=4\sqrt{2}+5\sqrt{2}=r_1+r_2$ ✓

Step 4: The point of contact divides AB internally in ratio $r_1:r_2$ = 4:5.

$$x = (4\times9 + 5\times0)/(4+5) = 36/9 = 4$$

$$y = (4\times(-1) + 5\times8)/(4+5) = (-4+40)/9 = 36/9 = 4$$

✅ **Point of contact = (4, 4)**

SECTION 10: REVIEW EXERCISE 13 — COMPLETE SOLUTIONS

Question 1 — Find centre and radius of each circle

NOTE: Method: Rewrite as $(x+g)^2+(y+f)^2 = g^2+f^2-c$. Centre=$(-g,-f)$, $r=\sqrt{(g^2+f^2-c)}$

(i) $x^2+y^2-4x+6y-12=0$

$g=-2, f=3, c=-12 \rightarrow$ Centre=$(2,-3)$, $r=\sqrt{(4+9+12)}=\sqrt{25}=5$

(ii) $x^2+y^2+4x-6y+4=0$

$g=2, f=-3, c=4 \rightarrow$ Centre=$(-2,3)$, $r=\sqrt{(4+9-4)}=3$

(iii) $x^2+y^2+6x-8y+16=0$

$g=3, f=-4, c=16 \rightarrow$ Centre=$(-3,4)$, $r=\sqrt{(9+16-16)}=3$

(iv) $x^2+y^2+x+4y-2=0$

$g=\frac{1}{2}, f=2, c=-2 \rightarrow$ Centre=$(-\frac{1}{2},-2)$, $r=\sqrt{(\frac{1}{4}+4+2)}=\sqrt{(25/4)}=5/2$

(v) $x^2+y^2+2x-4y+1=0$

$g=1, f=-2, c=1 \rightarrow$ Centre=$(-1,2)$, $r=\sqrt{(1+4-1)}=2$

(vi) $x^2+y^2-6x-10y+25=0$

$g=-3, f=-5, c=25 \rightarrow$ Centre=$(3,5)$, $r=\sqrt{(9+25-25)}=3$

(vii) $x^2+y^2+8x+2y-8=0$

$g=4, f=1, c=-8 \rightarrow$ Centre=$(-4,-1)$, $r=\sqrt{(16+1+8)}=5$

(viii) $x^2+y^2-16x-8y=64$

$g=-8, f=-4, c=-64 \rightarrow$ Centre=$(8,4)$, $r=\sqrt{(64+16+64)}=12$

(ix) $x^2+y^2+6x=0$

$g=3, f=0, c=0 \rightarrow$ Centre=$(-3,0)$, $r=3$

(x) $x^2+y^2+6x-4y+12=0$

$g=3, f=-2, c=12 \rightarrow$ Centre=$(-3,2)$, $r=\sqrt{(9+4-12)}=1$

(xi) $x^2+y^2+2x-8y+1=0$

$g=1, f=-4, c=1 \rightarrow$ Centre=$(-1,4)$, $r=4$

(xii) $x^2+y^2-6y=0$

$g=0, f=-3, c=0 \rightarrow$ Centre=$(0,3)$, $r=3$

(xiii) $x^2+y^2+2x=3 \rightarrow x^2+y^2+2x-3=0$

$g=1, f=0, c=-3 \rightarrow$ Centre=$(-1,0)$, $r=2$

(xiv) $x^2+y^2-6x-3=0$

$g=-3, f=0, c=-3 \rightarrow \text{Centre}=(3,0), r=2\sqrt{3}$

Question 2 — Find equations of circles through 3 points

(i) Through (2,0), (3,−3) and (0,−3)

Step 1: General form: $x^2+y^2+2gx+2fy+c=0$. Substitute each point.

Point (2,0): $4+4g+c=0 \rightarrow 4g+c=-4$ ......(1)

Point (3,−3): $9+9-6f+6g+c=0 \rightarrow 6g-6f+c=-18$ ......(2)

Point (0,−3): $9-6f+c=0 \rightarrow -6f+c=-9$ ......(3)

Step 2: (2)−(1): $2g-6f=-14 \rightarrow g-3f=-7$ ......(4)

Step 3: (1)−(3): $4g+6f=5$ ......(5)

Step 4: From (4): $g=3f-7$. Sub into (5): $4(3f-7)+6f=5 \rightarrow 18f=33 \rightarrow f=11/6$

$g = 3(11/6)-7 = 11/2-7 = -3/2$

$c = -4-4(-3/2) = -4+6 = 2$

✓ **Circle: $x^2+y^2-3x+(11/3)y+2=0$ (multiply by 3 for cleanliness)**

(ii) Through (3,4), (0,5) and (4,−3)

Step 1: Substitute each point into $x^2+y^2+2gx+2fy+c=0$:

Point (3,4): $25+6g+8f+c=0 \rightarrow 6g+8f+c=-25$ ......(1)

Point (0,5): $25+10f+c=0 \rightarrow 10f+c=-25$ ......(2)

Point (4,−3): $25+8g-6f+c=0 \rightarrow 8g-6f+c=-25$ ......(3)

Step 2: (1)−(2): $6g-2f=0 \rightarrow g=f/3$ ......(4)

Step 3: (3)−(2): $8g-16f=0 \rightarrow g=2f$ ......(5)

Step 4: From (4)&(5): $f/3=2f \rightarrow f=0, g=0$. From (2): $c=-25$

✓ **Circle: $x^2+y^2-25=0$ (centre origin, radius 5)**

Question 3 — Circle with diameter endpoints

NOTE: If (x_1,y_1) and (x_2,y_2) are ends of diameter: $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

(i) (1,−1) and (4,1)

$(x-1)(x-4)+(y+1)(y-1)=0$

$x^2-5x+4+y^2-1=0$

✓ **$x^2+y^2-5x+3=0$**

(ii) (3,−5) and (2,0)

$(x-3)(x-2)+(y+5)(y)=0$

$x^2-5x+6+y^2+5y=0$

✓ **$x^2+y^2-5x+5y+6=0$**

(iii) (−2,−1) and (3,2)

$(x+2)(x-3)+(y+1)(y-2)=0$

$x^2-x-6+y^2-y-2=0$

✓ **$x^2+y^2-x-y-8=0$**

Question 4 — Points P(2,4), Q(8,−2), R(6,2)

(a) Equation of line l perpendicular to PQ through midpoint of PR

Step 1: Midpoint of PR = ((2+6)/2, (4+2)/2) = (4, 3)

Step 2: Gradient of PQ = (−2−4)/(8−2) = −1

Step 3: Gradient of l = 1 (perpendicular). Line through (4,3): y−3=1(x−4)

✓ **l: y = x − 1 or x − y − 1 = 0**

(b) Centre C and radius of circle through P, Q, R

Step 1: Centre lies on perpendicular bisectors. Find perp bisector of PQ too.

Step 2: Midpoint of PQ = (5,1). Gradient of PQ=−1, so perp bisector gradient=1.

Perp bisector of PQ: $y-1=1(x-5) \rightarrow y=x-4$

Step 3: Intersect with l: x−1=x−4 — parallel! Use perp bisector of QR instead.

Step 4: Midpoint QR=((8+6)/2,(−2+2)/2)=(7,0). Gradient QR=(2+2)/(6−8)=−2.

Perp bisector QR: $y-0 = \frac{1}{2}(x-7) \rightarrow y=x/2-7/2$

Step 5: Intersect perp bisector PQ (y=x−4) with perp bisector QR (y=x/2−7/2):

$x-4 = x/2-7/2 \rightarrow x/2=1/2 \rightarrow x=1, y=-3$

Centre $C=(1,-3)$, $r=\sqrt{[(2-1)^2+(4+3)^2]}=\sqrt{50}=5\sqrt{2}$

✓ **Centre C = (1, −3), Radius = $5\sqrt{2}$**

Question 5 — Circle through O(0,0), A(3,3), B(3,1)

Step 1: Substitute into $x^2+y^2+2gx+2fy+c=0$.

$O(0,0)$: $c=0$

$A(3,3)$: $18+6g+6f=0 \rightarrow g+f=-3$ ......(1)

$B(3,1)$: $10+6g+2f=0 \rightarrow 3g+f=-5$ ......(2)

Step 2: (2)−(1): 2g=−2 → g=−1, f=−2

Step 3: Equation: $x^2+y^2-2x-4y=0 \rightarrow (x-1)^2+(y-2)^2=5$

Centre=(1,2), $r=\sqrt{5}$ ✓

Step 4: Verify P(2,4): $(2-1)^2+(4-2)^2=1+4=5=r^2$ ✓ P lies on circle.

Step 5: Gradient of tangent at P: radius CP gradient = (4−2)/(2−1) = 2, so tangent gradient = −½

Tangent at P(2,4): $y-4=-\frac{1}{2}(x-2) \rightarrow 2y-8=-x+2 \rightarrow x+2y=10$

✓ **Circle: $x^2+y^2-2x-4y=0$. Tangent at P(2,4): x+2y=10**

Question 6 — Circle $x^2+y^2+14x-20y+129=0$

Step 1: Complete the square: $(x+7)^2+(y-10)^2=49+100-129=20$

$\text{Centre}=(-7,10),\ r=\sqrt{20}=2\sqrt{5}$

Step 2: Verify P(−3,12): $(-3+7)^2+(12-10)^2=16+4=20=r^2$ ✓

Step 3: Gradient of CP = (12−10)/(−3+7) = 2/4 = ½

Step 4: Tangent gradient = −2 (perpendicular). Tangent at P(−3,12):

$y-12=-2(x+3) \rightarrow y=-2x+6 \rightarrow y=6-2x$

✅ (a) Centre=(−7,10), $r=2\sqrt{5}$. (b) Tangent at P: y=6−2x ✓

QUICK REFERENCE — KEY STEPS SUMMARY

Centre & radius	Rewrite as $(x+g)^2+(y+f)^2=r^2$. Centre=(−g,−f), $r=\sqrt{(g^2+f^2-c)}$
Tangent length	$\sqrt{(CP^2-r^2)}$ where CP = dist from external point to centre
Line is tangent?	Perpendicular distance from centre to line = r
Orthogonal	$AB^2 = r_1{}^2 + r_2{}^2$
Touch externally	$AB = r_1 + r_2$
Touch internally	$AB = \lvert r_1 - r_2 \rvert$
Locus	Let P=(x,y), write geometric condition, form equation, simplify
Intersection	Substitute line into circle equation, solve simultaneously
Point on curve?	Substitute coordinates — if LHS=0, point lies on curve
Tangent at point	Radius ⊥ tangent. Find gradient of radius, take negative reciprocal

Master the pattern — every circle question follows these same steps!